1501ENG Engineering Mechanics

Trimester 2, 2023

Supplementary Material & Workbook

(Weeks 1 to 12)

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CONTENTS

PART 1 – LECTURE NOTES

CHAPTER 1 FUNDAMENTALS

OBJECTIVES AND EXPECTED OUTCOMES

- *Understand the concepts of forces, moments and rigid bodies for statics*
- *Distinguish between beams, two-force members, columns and identify such components in actual structures*
- *Distinguish between concentrated loads, UDL, LDL*
- *Sketch centre-line diagram (analytical model) of basic engineering structures*

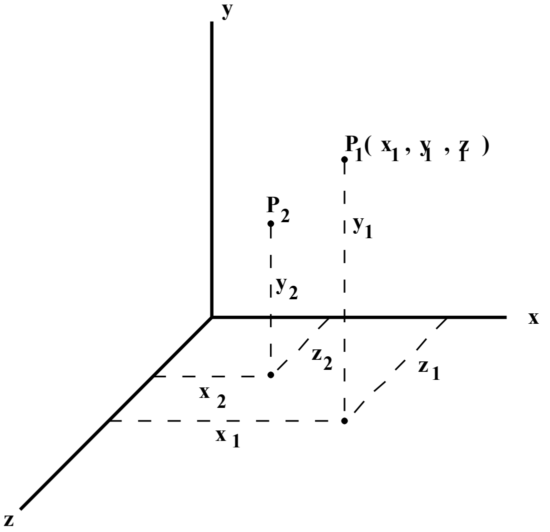
1.1 GENERAL REMARKS

- Classical mechanics - statics and dynamics originated from three fundamental Newton's **laws of motion**.
- Statics – the branch of mechanics that is concerned with the analysis of loads (forces and moments) on physical systems in static equilibrium.
- Applications of three **laws** in analysing engineering related phenomena and in providing solutions to problems encountered by engineers are unlimited.

1.2 BASIC CONCEPTS

Space

- Together with time space is sometimes conceived as containers in which all events and processes occur.
- In 3D reference system x, y and z, the **location** of a given point P can be described.
- **Size** and **shape** of an object/structure can be defined with the same reference system.
- In majority of engineering problems, the effects in a third dimension may be ignored. Then 3D space may be reduced to a 2D (or planar) reference system.



Rectangular reference system

Time

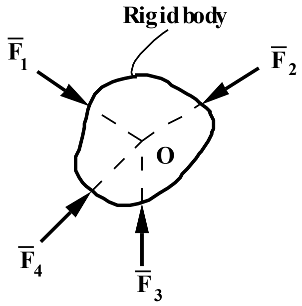
- Time has no beginning and it has no end.
- Space is three dimensional, time is of one dimension.
- In statics, time is irrelevant.
- In dynamics time is measured from a reference zero point - any convenient instant.

Mass

- Fundamental property of a body or object.
- A measure of the body's resistance to any change to its motion.
- Two bodies of the same mass and the same distribution are attracted by the earth in the same manner.

Force

- A force may be exerted by contact or at a distance.
- A pull, a push or the weight of an object is an example of a force which is characterised by its **magnitude**, **point of application** and **direction**.



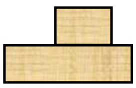
- The point of action and the direction of the force together define the **line of action**. Force is a **vector**.

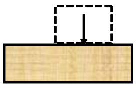
1.3 NEWTON'S LAWS OF MOTION

Law 1 A particle remains at rest or continues in uniform motion in a straightline unless it is acted upon by a resultant force.

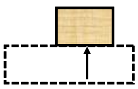
Law 2 A particle acted upon by a resultant force receives an acceleration in the direction of the force that is proportional to the force and inversely proportional to the mass of the particle.

Law 3 For every action upon a particle, it exerts an equal, opposite, and collinear reaction.





Force of small block on large block



Equal and opposite reaction force from large block on small block

- **Statics** = no Motion
- **Dynamics** = Motion
- Newton's Laws of Motion apply to statics simply by setting the accelerations to zero

1.4 UNITS

Standard **S.I.** system of units:

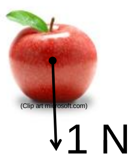
Length: m or mm, cm
Area: m^2, mm^2
Volume: m^3, mm^3
Time: s (second)
Mass: kg
$\boxed{1 \text{ tonne} = 1000 \text{ kg} = 10^3 \text{ kg}}$

Force: N (Newton)
$1 \text{ N} = 1 \text{ kg m/s}^2$
$\boxed{1 \text{ kN} = 10^3 \text{ N}}$

Weight (Force) = mg (units: N)

$g = 9.80 \text{ m/s}^2$

How heavy is 1 N? — A small apple. (10 small apples ≈ 1 kg)



1.5 RIGID BODY

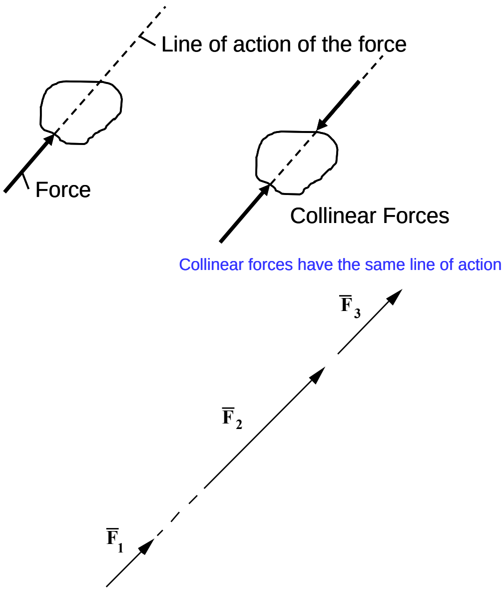
- 'Particle' in Newton's laws of motion may be generalised to include a **non-deformable** body - a **rigid body**.
- **Rigid** body may be deformed under external forces so long as the deformation is 'microscopic' or so small that it does not affect the application of laws of motion.
- An extended object whose size and shape do not change as it moves or as it is loaded.
- In Statics, the objects concerned may be considered **rigid**. (steel beam, house, bridge and its individual members).


1.6 COLLINEAR, CONCURRENT AND NONCONCURRENT FORCES

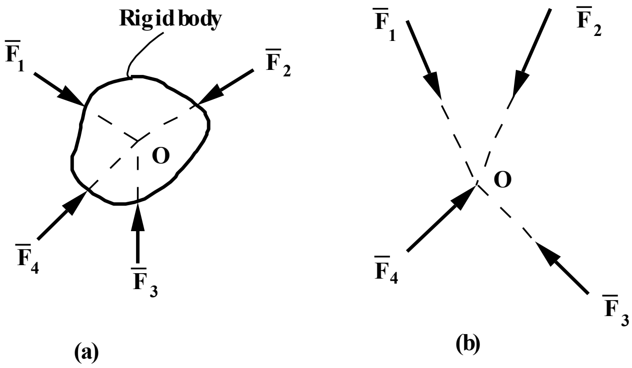
- **Collinear forces** – two or more forces whose lines of action coincide with each other
- Tug-of-war, action and reaction

- **Concurrent force** system – two or more forces whose lines of action intersect at one point
- Collinear forces are a particular case of concurrent forces

- **Nonconcurrent force** system – If such a concurrent point does not exist

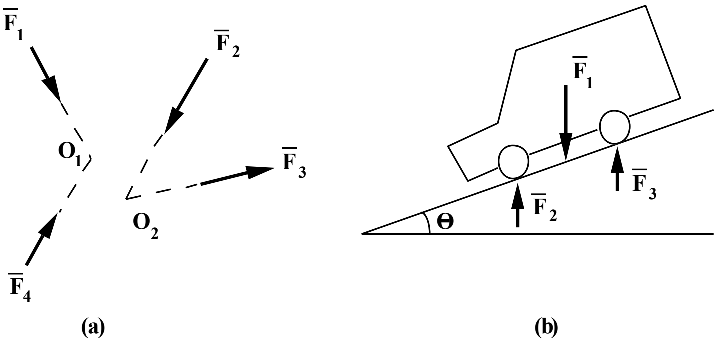


Collinear forces



‘concurrent’ means ‘same point’

Concurrent forces



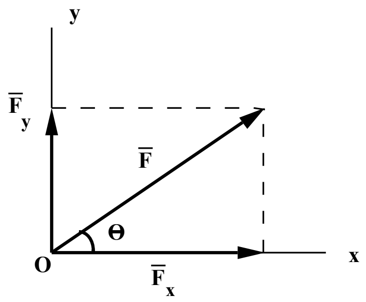
Nonconcurrent forces

1.7 COMPONENT FORCE

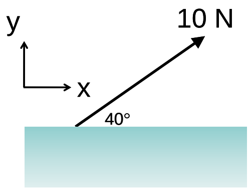
1.7.1 Concepts

Force is a vector

- magnitude
- direction
- point of application
- can be resolved into two or more component forces



Force and its components



x-component of 10N?

(a) $10\,sin\,40°$
(b) $10\,cos\,40°$
(c) $-10\,cos\,40°$
(d) $10/sin\,40°$

y-component of 10N?

(a) $10\,sin\,40°$
(b) $10\,cos\,40°$
(c) $-10\,cos\,40°$
(d) $10/sin\,40°$

Two components $\overline{F}_x$ and $\overline{F}_y$ parallel to the x and y-axes, respectively

$$F_X = F\cos\theta = F \times x/r \qquad ...\ (1.1)$$

$$F_Y = F\sin\theta = F \times y/r \qquad ...\ (1.2)$$

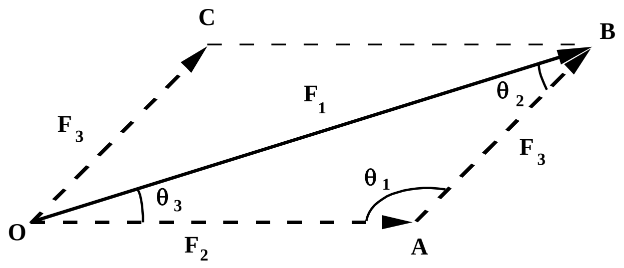
Note: bar on top of a symbol representing a vector will be dropped for convenience.

- **Parallelogram method**

- **Triangular method**

1.7.2 Laws of Sine and Cosine (algebraic method to determine component forces)

Resolve a force into components in other directions not parallel to the x and y axes:



Law of sine

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3} \qquad \text{. . . (1.3)}$$

- Relative directions of F_1, F_2 and F_3 are θ_1, θ_2 and θ_3 ($\theta_1 + \theta_2 + \theta_3 = 180°$).
- Point of application of F_3 is O.

Law of cosine

- Magnitude of any one of the three forces may be determined if its subtended angle and the other two force magnitudes are known:

$$F_1^2 = F_2^2 + F_3^2 - 2F_2F_3\cos\theta_1 \qquad (\text{if } \theta_1\text{=90°, } F_1^2 = F_2^2 + F_3^2) \qquad \text{. . . (1.4)}$$

$$F_2^2 = F_1^2 + F_3^2 - 2F_1F_3\cos\theta_2 \qquad \text{. . . (1.5)}$$

and

$$F_3^2 = F_1^2 + F_2^2 - 2F_1F_2\cos\theta_3 \qquad \text{. . . (1.6)}$$

1.7.3 General Case in 2-D Plane

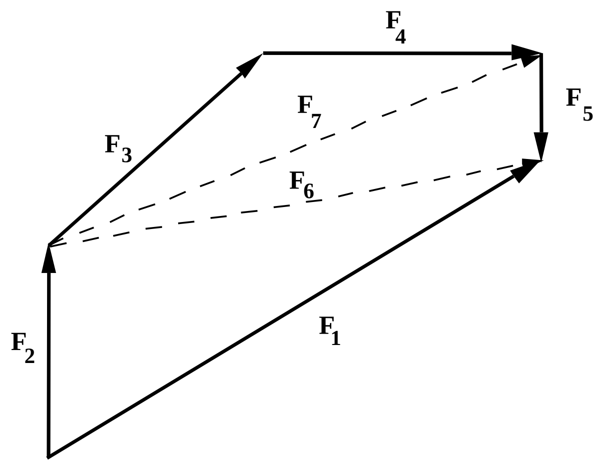
Resolution of a force into more than two components: F_1 is resolved into F_2, F_3, F_4 and F_5 in the directions shown.

Steps:

(1) Resolve F_1 into F_2 and F_6,

(2) Resolve F_6 into F_7 and F_5.

(3) Resolve F_7 into F_3 and F_4.



1.7.4 Comments

A problem (of a force and its two components) is defined or has a solution if:

(i) the magnitude of one of the three forces and the relative directions of the remaining two forces are known: **sine rules**

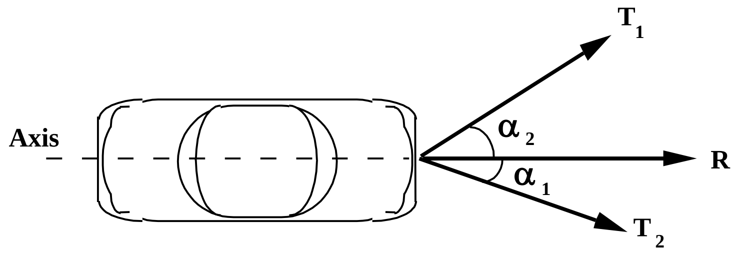
(ii) the magnitudes of two of the three forces are known together with the subtended angle of the unknown force: **cosine rules**

1.7.5 Illustrative Example

A disabled car is towed by means of two ropes as shown below. If the force R is 1500N parallel to the axis of the car, determine

(i) the tension in each of the ropes, i.e. T_1 and T_2 if $\alpha_1 = 20^o$ and $\alpha_2 = 30^o$;

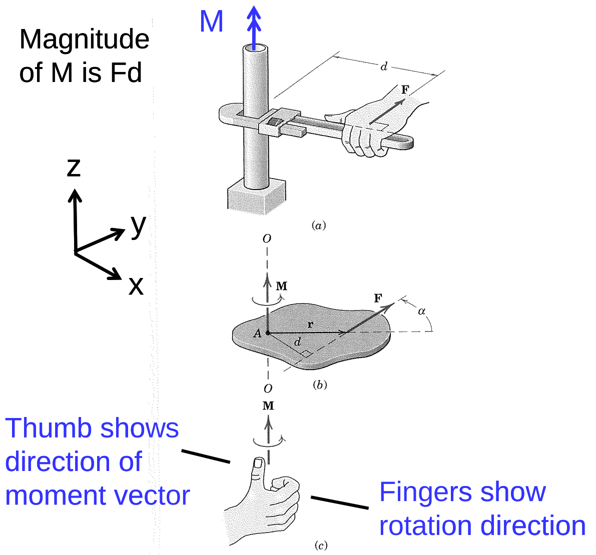
(ii) the value of α_2 such that T_1 is a minimum (where α_1 remains 20^o)



SOLUTION

1.8 MOMENT

1.8.1 Concepts



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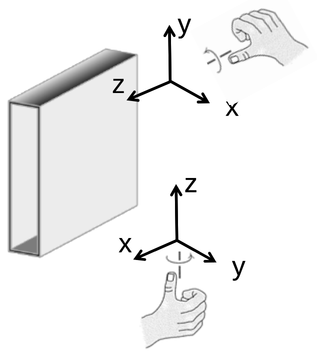


Right hand coordinate systems

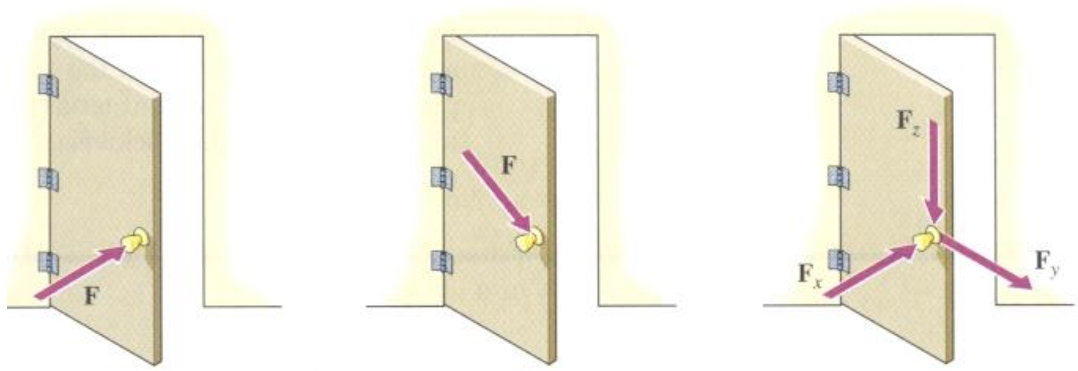
By the right-hand rule an anti-clockwise rotation is positive

- **A <u>moment</u> is the tendency of a force to cause a rotation about a point or axis which in turn produces bending stresses**

- **A <u>moment</u> is a twisting action; a torque; a force acting at a distance from a point in a structure so as to cause a tendency of the structure to rotate about that point**

- **A <u>moment</u> is calculated by multiplying the magnitude of a force by the length of its lever arm, the perpendicular distance between the line of action of the force and the point where it is applied**

- **A <u>moment</u> of a couple is the product of its force and the perpendicular distance between its opposing forces**



Definition: Moment M, of a force F about a point O (Fig. 1.10) is defined as

$$M = Fd \qquad \ldots(1.7)$$

- d = the lever arm, is the shortest distance between O and the line of action of F.
- Moment is denoted by symbol ↻ for a clockwise, and ↺ for an anticlockwise one.
- The point indicates the point of action of the moment and the circular arrow, its direction.
- By definition, moment is the product of force and distance. Unit – kNm, Nm or Nmm.

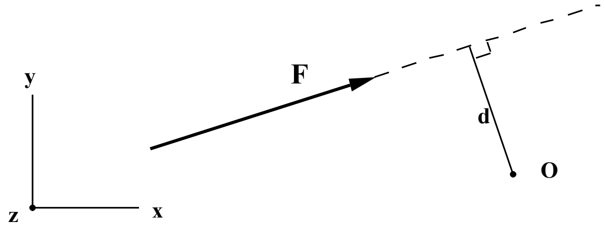


Fig. 1.10

- F is acting in a plane x-y
- M about O is acting about an axis which passes through O and is parallel to the z axis
- F produces no moment about the x or y axis since its extension intersects with these two axes.

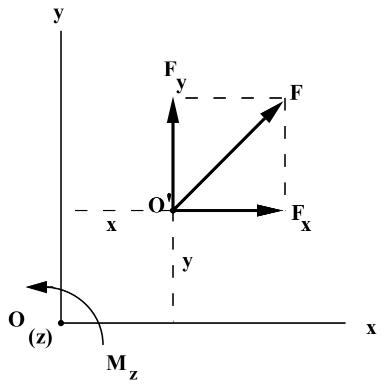


Fig. 1.11

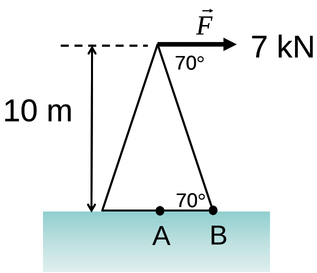
$M_x = 0$

$M_y = 0$

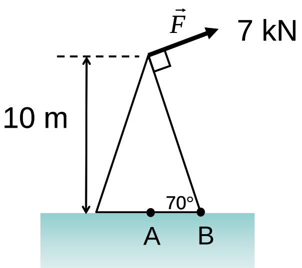
$$M_z = F_y \cdot x - F_x \cdot y \qquad \ldots(1.8)$$

Examples

Which moment has a greater magnitude?



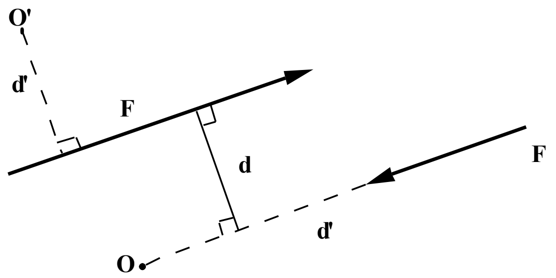
(a) The moment of the force F about A
(b) The moment of the force F about B
(c) They both have the same magnitude



(a) The moment of the force F about A
(b) The moment of the force F about B
(c) They both have the same magnitude

1.8.2 Moment of a Couple

- **Couple** — A pair of parallel, noncollinear forces having the same magnitude but acting in opposite directions.
- Moment produced by a couple of forces is M=Fd and it is constant with respect to any point in the plane of application.



Moment about point O (assuming clockwise direction +Ve):

$$M_O = F{\cdot}d + F{\cdot}(0) = Fd\ (\circlearrowright)$$

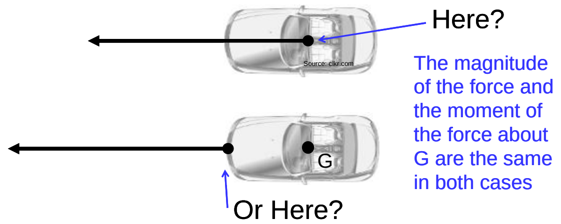
Moment about an arbitrary point O' :

$$M_{O'} = F\,(d + d') - Fd'$$
$$= Fd\ (\circlearrowright) = M_O \ \ (\textbf{constant})$$

1.8.3 Lateral Transfer of a Force (*Is the point of action of the force important?*)

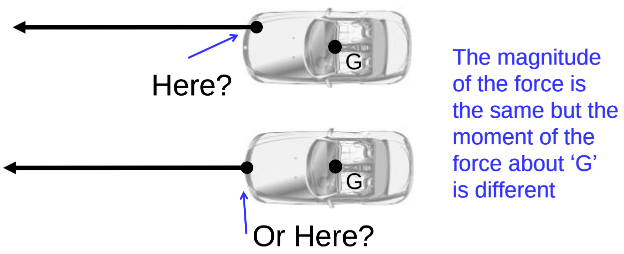
- Newton's laws of motion — movement of a rigid body is determined by the magnitude and direction of the force **ONLY**, but not the point of application.
- Points of application may translate along the line of action without altering the effects on the rigid body's movement.

  Will it make any difference if we tie a rope and apply the same force?

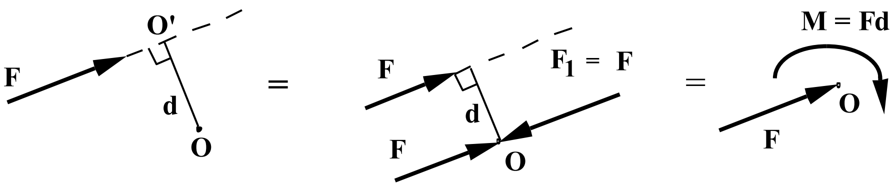


- If the force is transferred laterally, the effects will change.

  Will it make any difference if we tie a rope and apply the same force?



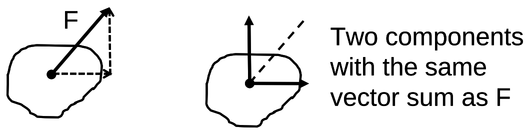
- So that the effects on rigid body movement are unchanged, some compensation in the form of a moment must be made.



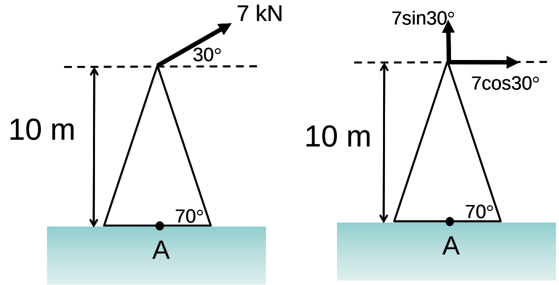
(a) (b) (c)

1.8.4 Comments

- Force and components – The Force may be replaced by components that have the same vector sum as the original force

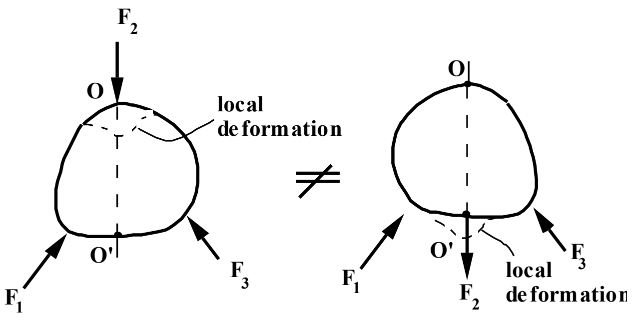


Original Force   Equivalent system

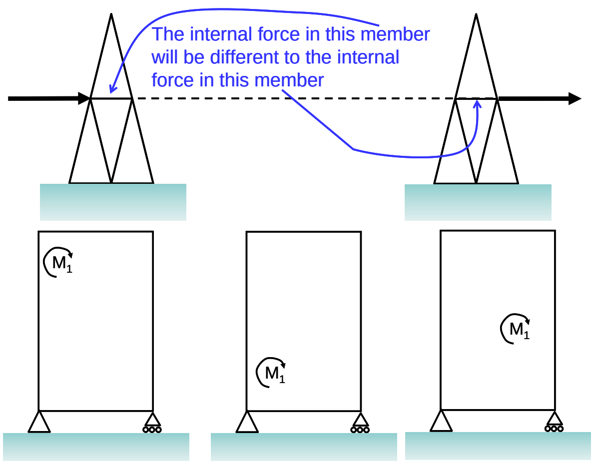


- Moment – The moment of a force about any point is equal to the sum of the moments of the components of the force about the same point

- Rigid body – **non-deformable**, points of application may translate along the line of action without altering the effects on the rigid body's movement**.**
- When dealing with the internal forces or the deformation of a structure (an elastic body - **deformable**), the point of application of a force cannot be altered.



Deformation of an elastic body



External reaction forces are the same, but internal stress (i.e. internal force) distribution will change!

1.9 TYPES OF STRUCTURES AND STRUCTURAL ELEMENTS

1.10 TYPES OF LOADING

1.11 STRUCTURAL ANALYSIS

- **Internal forces** – constituent members of a structure under loads develop forces and bending moments;
- **Support reactions** – reaction forces and moments are produced by applied loading;
- **Structural analysis** – computational process leading to the magnitude and directions of internal forces and support reactions.

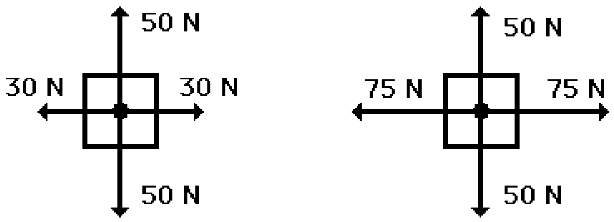
CHAPTER 2 EQUILIBRIUM

OBJECTIVES AND EXPECTED OUTCOMES

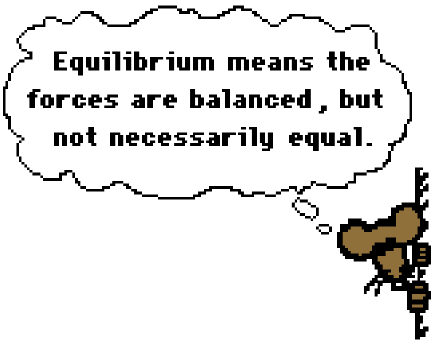
- *Understand the origin and assumptions of the equilibrium equations*
- *Apply the equilibrium equations to solve:*
  - *problems involving concurrent forces*
  - *problems involving 2D rigid bodies*

2.1 GENERAL

- **Concept of equilibrium of forces is derived from the 1st law of motions.**
- **When all the forces which act upon an object are balanced, then the object is said to be in a state of equilibrium.**
- **The forces are considered to be balanced if the rightward forces are balanced by the leftward forces and the upward forces are balanced by the downward forces.**
- **This however does not necessarily mean that the forces are *equal*.**



These two objects are at equilibrium since the forces are balanced. However, the forces are not equal.



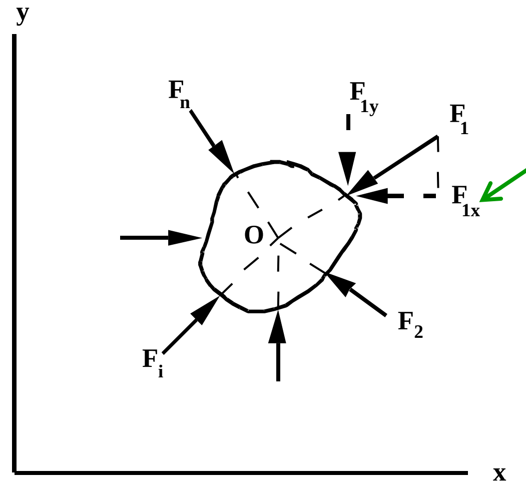
An object at equilibrium is either ...

- **at rest and staying at rest , or**
- **in motion and continuing in motion with the same speed and direction.**

Statics is the branch of mechanics dealing with physical systems that are in static equilibrium

2.2 EQUILIBRIUM IN 2-D PLANE

2.2.1 Concurrent Forces



(Here F_{1x} has a negative value)

$$F_{1x} + F_{2x} + F_{3x} + \ldots + F_{ix} + \ldots + F_{nx} = 0 \quad (2.1)$$

$$F_{1y} + F_{2y} + F_{3y} + \ldots + F_{iy} + \ldots + F_{ny} = 0 \quad (2.2)$$

Another way to write this is:

$$\sum F_x = 0 \quad (2.3)$$

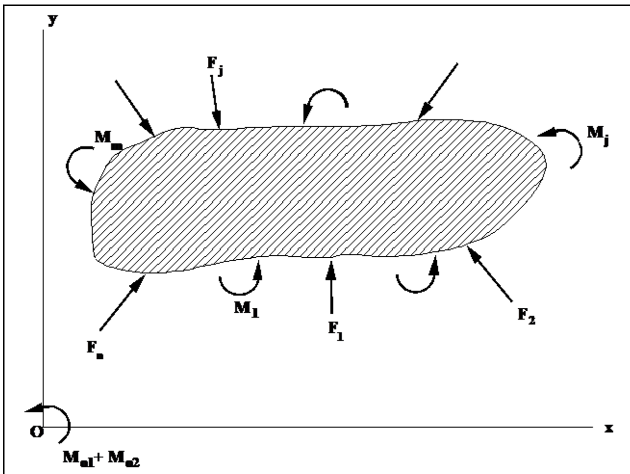
$$\sum F_y = 0 \quad (2.4)$$

2 eqs. must be satisfied simultaneously

Case of **concurrent forces** – don't need to worry about moments because all forces pass through the same point ($\Sigma M_O = 0$)

- A set of the concurrent forces acting at various points of a rigid body (truss, beam etc).
- The rigid body is at rest indicating that the forces F_1, F_2...............F_n are in equilibrium.
- Does not move horizontally because the resultant in the horizontal direction is zero.
- Does not move vertically because the vertical component of the resultant is also zero.

2.2.2 Non-concurrent Forces



Rigid body at rest

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\Sigma M_O = 0$$

3 eqs. must be satisfied simultaneously

Moments are assumed to be anticlockwise which is positive following the right-hand screw rule.

$$M_1 + M_2 + \ldots + M_m +$$
$$F_{1x}y_1 + F_{2x}y_2 + \ldots + F_{nx}y_n +$$
$$F_{1y}x_1 + F_{2y}x_2 + \ldots + F_{ny}x_n = 0$$

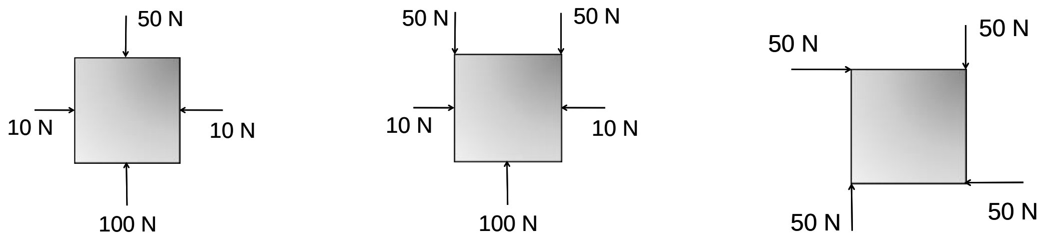
(Some F_{ix} and F_{iy} have negative values)

Another way to write this is:

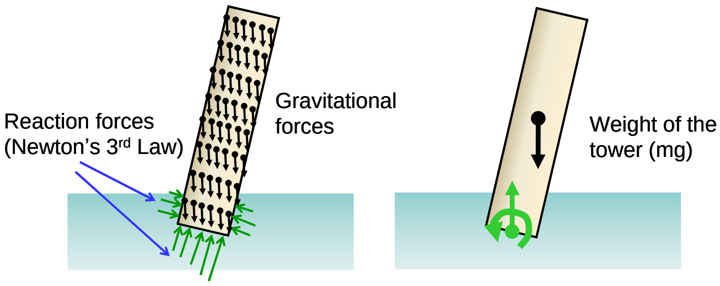
$$\sum_{i=1}^{n}\left(F_{ix}y_i + F_{iy}x_i\right) + \sum_{j=1}^{m} M_j = 0$$

Examples

Example 2.1 Are the following objects in equilibrium?



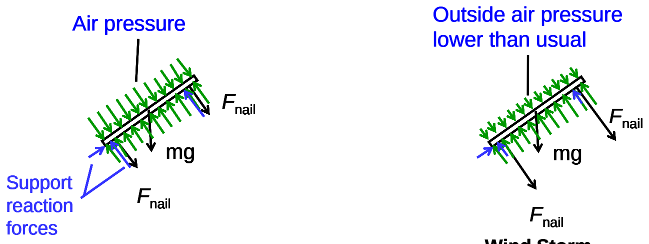
Example 2.2. The leaning Tower of Pisa



Forces acting on the tower

Simplified model

Example 2.3 Roof sheet in a wind storm

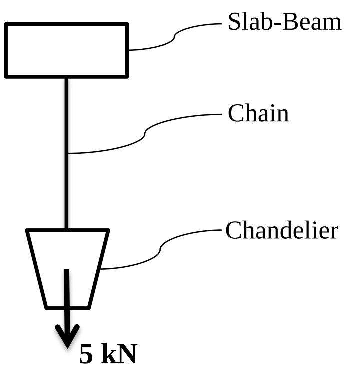


Usual situation
(Pressure inside and outside cancel)

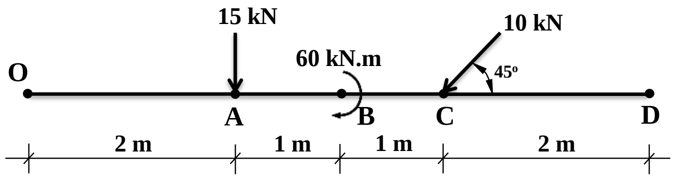
Wind Storm

Example 2.4 For each of the following planar systems, set up equilibrium equations $\sum F_x = 0$, $\sum F_y = 0$ and $\Sigma M_O = 0$.

(1) Slab-beam and chandelier



(2) A beam under concentrated forces and moment



CHAPTER 3 SUPPORT REACTIONS AND FREE BODY DIAGRAMS

OBJECTIVES AND EXPECTED OUTCOMES

- *Determine structural stability and determinacy*
- *Distinguish different types of supports and determine support reactions*
- *Understand the principle of superposition and free body diagram concepts*

3.1 THE ORIGIN OF SUPPORT REACTIONS

- Supports provide physical connections between the structure and its foundation.
- Supports ensure stability of the structure if properly designed.
- Supports serve to transfer the loads carried by the structure to the foundation.
- Supports provide physical restraints to the structure from movements in all directions and about any axis.
- Support reactions and applied forces maintain equilibrium.
- Equilibrium equations used to compute the magnitudes and directions of the support reactions.

3.2 TYPES OF SUPPORTS

3.3 STATICALLY DETERMINATE AND INDETERMINATE SUPPORT CONDITIONS

3.3.1 General Remarks

Three equilibrium equations must be satisfied for a nonconcurrent co-planar force system:

$$\sum F_x = 0 \qquad \text{. . . (3.1)}$$
$$\sum F_y = 0 \qquad \text{. . .(3.2)}$$
$$\sum M = 0 \qquad \text{. . . (3.3)}$$

Three equations applied to the rigid body as a whole → can only solve for **three** unknown external reactions.

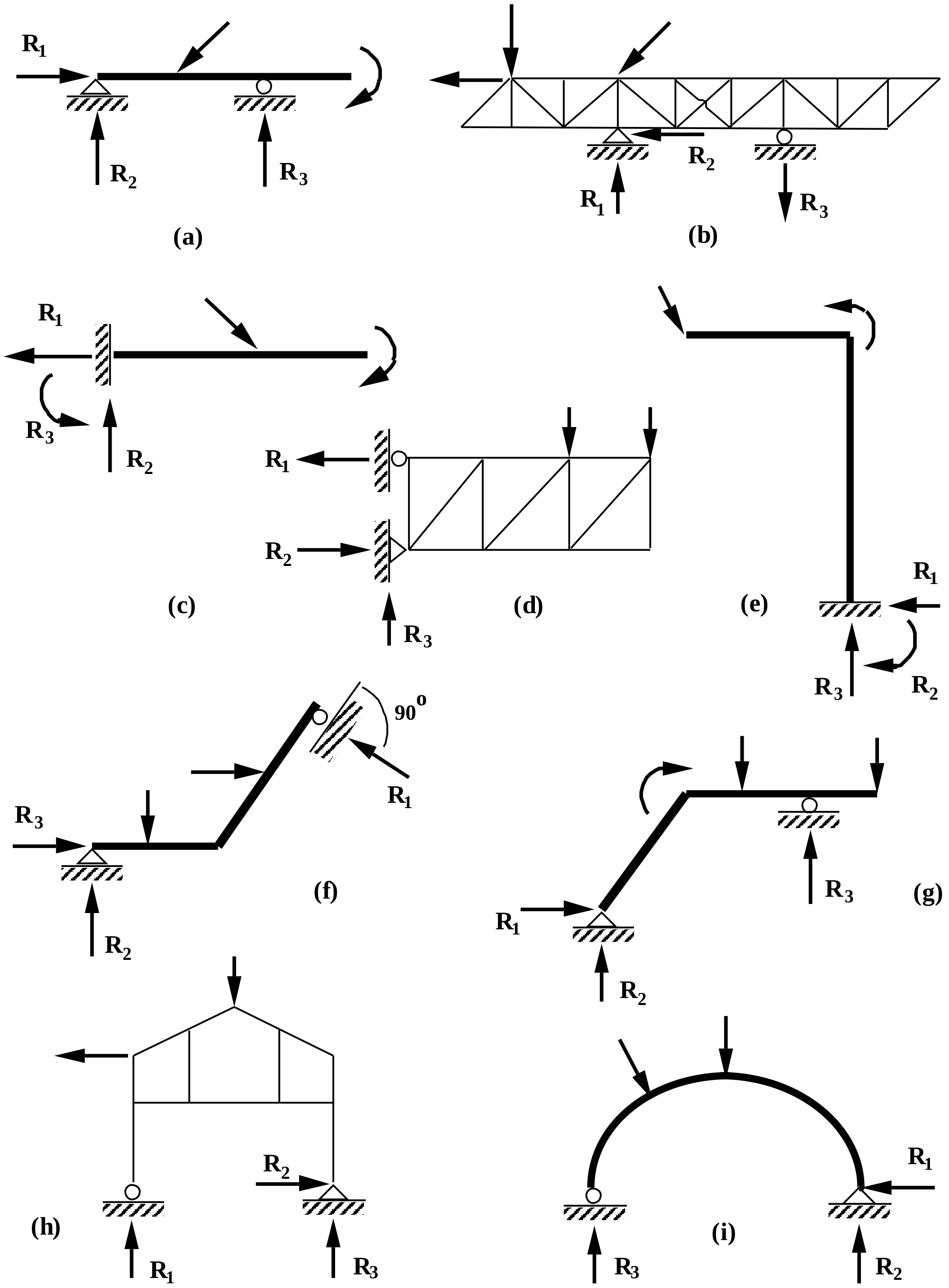
3.3.2 External Determinacy and Indeterminacy

Three independent reactions

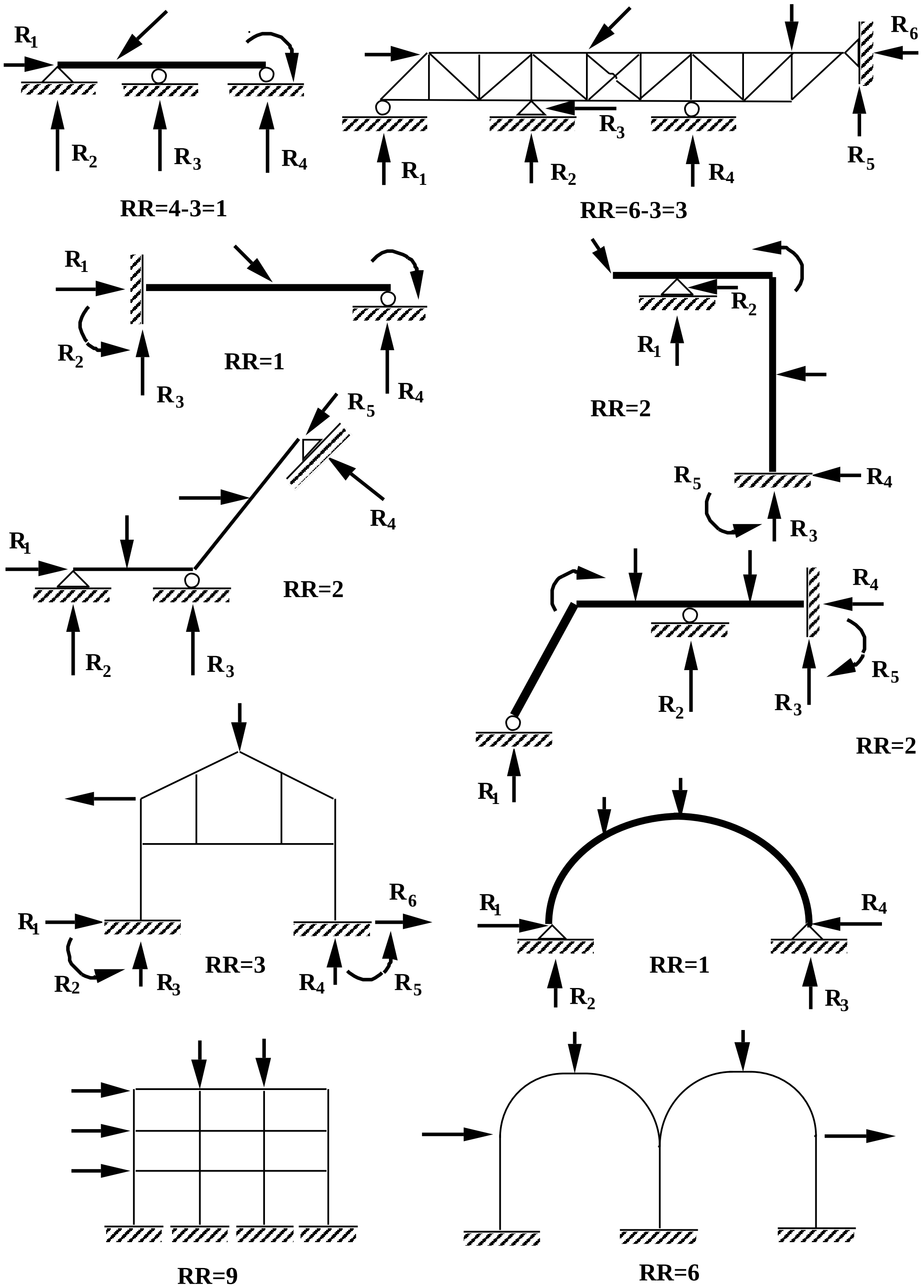
R = 3 (External Determinate, or SD) . . .(3.4)

More than **three** independent reactions

R > 3 (External Indeterminate, or SI) . . .(3.5)



Determinate support conditions (R = 3)



RR = Number of Redundant Reactions

Indeterminate support conditions (R>3)

- "**Redundant reactions**" - not needed to provide adequate support or minimum stability.
- In practice, redundant reactions are either unavoidable or they are deliberately engineered to **provide extra reserved strength** to the system.
- Support conditions of most modern structures are statically indeterminate.

3.3.3 Simply-Supported and Cantilever Structures

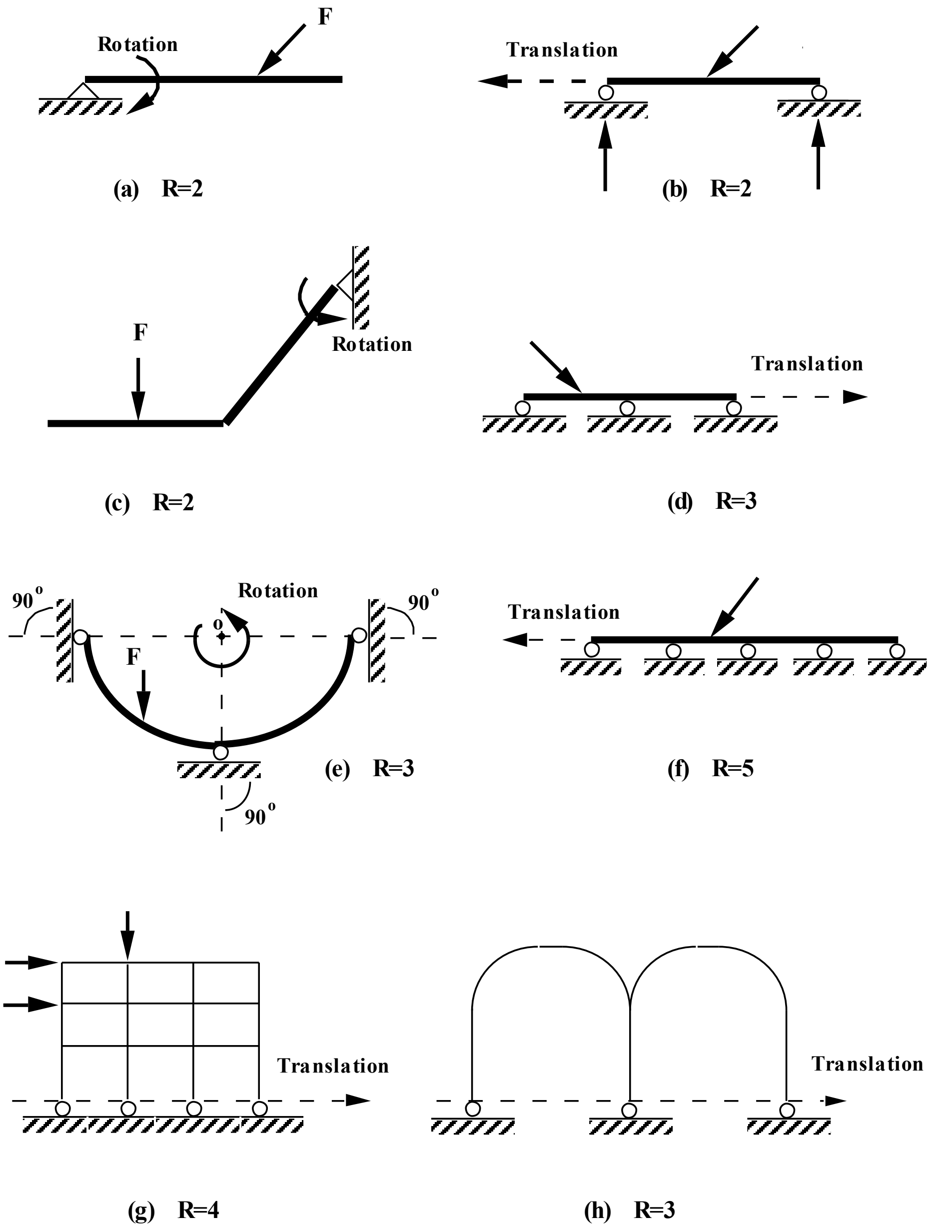
- **Simply-supported** structure **–** supported by a hinge and a roller (beam/truss/frame/arch).
- **Simply-supported with overhang(s) –** if either or both of the supports are not located at the extremities or ends.
- **Cantilever beam –** with a built-in support.
- **Cantilever truss –** overhanging truss supported by a roller and a hinge.

3.3.4 Stable and Unstable Support Conditions

A structure (excluding a two-force member) is inadequately supported or unstable if

$R < 3$ (Always unstable, **necessary and sufficient conditions**) . . .(3.6)

$R \geq 3$ (May or may not be stable, **necessary but insufficient conditions**)
(**Engineering inspection**: (a) reactions parallel?
(b) reactions meet at one point?)

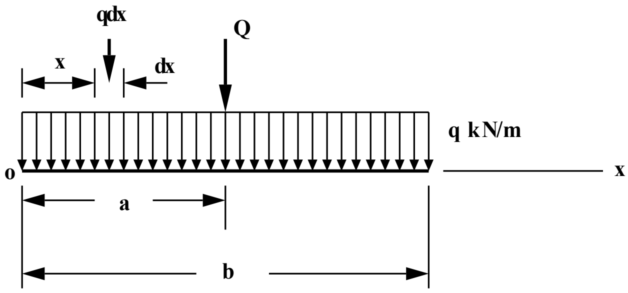


Unstable support conditions

3.4 DISTRIBUTED LOADS AND THEIR RESULTANTS

- Compute support reactions under distributed load
- Determine magnitude and location of the resultant

3.4.1 Uniformly Distributed Load (UDL)



- UDL acting over a distance b
- Resultant = Area under load curve

$$Q = \int_0^b q \cdot dx = [qx]_0^b = q \cdot b \qquad . . . (3.7)$$

- Moment produced by Q = that by UDL
  Taking moment about o,

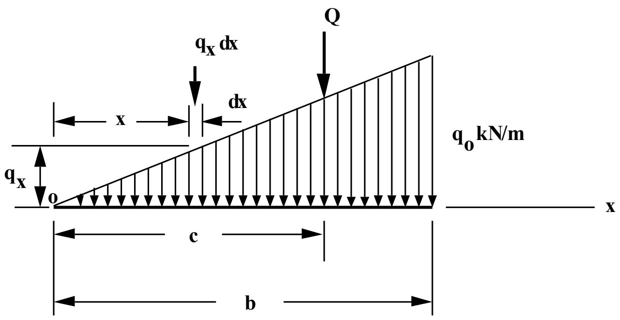
$$Q \cdot a = \int_0^b (q \cdot dx) \cdot x = \left[\frac{qx^2}{2}\right]_0^b = \frac{qb^2}{2} \qquad . . . (3.8)$$

- Location of Q

$$a = \frac{b}{2} \qquad . . . (3.9)$$

- Direction of Q = direction of q

3.4.2 Linearly Distributed Loads (LDL)



- LDL acting over a distance b
- Resultant = Area under load curve

$$q_x = \frac{x}{b} \cdot q_o \qquad . . . (3.10)$$

$$Q = \int_0^b q_x \cdot dx = \int_0^b \frac{q_o x}{b} \cdot dx = \left[\frac{q_o x^2}{2b}\right]_0^b = \frac{q_o b}{2} \qquad . . . (3.11)$$

- Moment produced by Q = that by LDL
  Taking moment about o,

$$Q \cdot c = \int_0^b (q_x \cdot dx) \cdot x = \int_0^b \frac{q_o x^2}{b} \cdot dx = \frac{q_o b^2}{3}$$

$$\frac{(q_o b) \cdot c}{2} = \frac{q_o b^2}{3}$$

- Location of Q

$$c = \frac{2b}{3} \qquad . . . (3.12)$$

- Direction of Q = direction of q

3.4.3 Comments

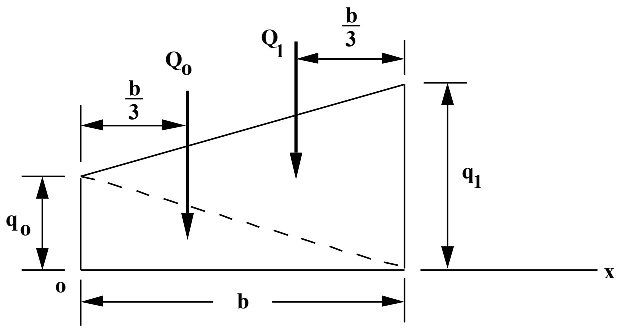
UDL:

- Q = Area of rectangle
- Q acts through centre of the rectangle
- Direction of Q is the same as that of UDL

LDL:

- Q = Area of triangle
- Q acts through centre of the triangle
- Direction of Q is the same as that of LDL

Same principle is applicable to a distributed load having any shape, **trapesoidal**, **parabolic –** the resultant is the area and its acts through the centre of the area.



- LDL in a trapezoid form
- Area can be represented by two triangles
- Two resultants

$$Q_o = \frac{q_o b}{2}$$ Location: b/3 from its base . . . (3.13)

and, $$Q_1 = \frac{q_1 b}{2}$$ Location: b/3 from its base . . . (3.14)

- Total resultant

$Q = Q_o + Q_1$

- More complicated loads can be broken up into more than one resultant
- Replacing the distributed load with a **resultant force** Q is only valid for finding the support reactions.
- It is not applicable for determining internal forces or for studying the deformation of the structure. In these cases the **actual loads** must be used.

3.5 REACTIONS FOR PLANAR STRUCTURES

Analysis procedure:

(1) Ensure that the given structure is stable and S.D.

(2) Assume the directions of all the reactions.

(3) Set up the relevant equilibrium equations based on these assumed directions.

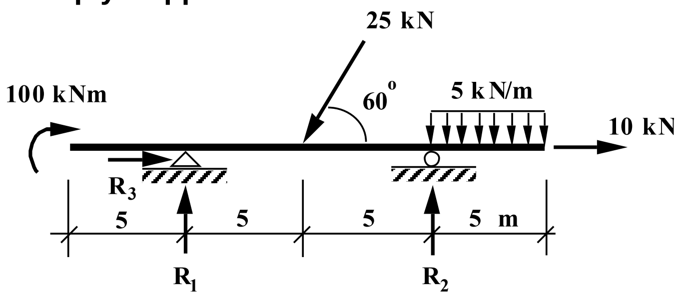
- $\Sigma F_X = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$
- Assume +Ve direction and be consistent
- Must include all forces and moments
- Take moment about a convenient point (e.g. hinge support), so that 2 out of 3 unknowns will be eliminated

(4) Solve equilibrium equations to obtain reactions

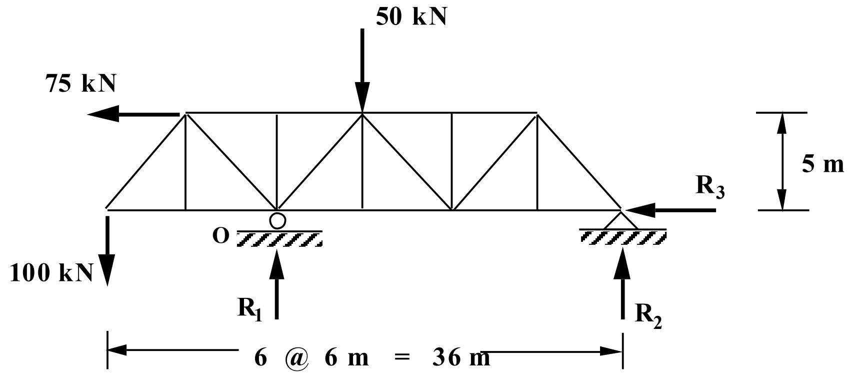
- +Ve answer → assumed direction is correct
- - Ve answer → actual direction should be opposite to the original assumption

(5) Do a static check

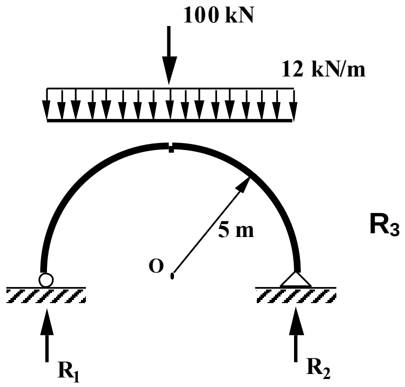
3.5.1 Simply-Supported Beams



3.5.2 Simply-Supported Trusses

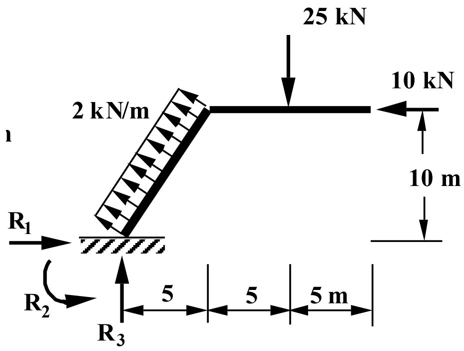


3.5.3 Simply-Supported Frames and Arches



Example 3.1 Compute the reactions for the simply-supported arch.

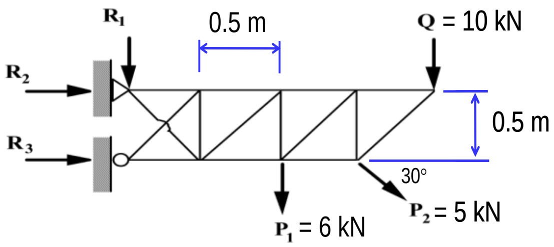
3.5.4 Cantilever Beams and Frames



Example 3.2 Compute the reactions R_1, R_2 and R_3 for the cantilever frame.

3.5.5 Cantilever Trusses

Compute the reaction forces for the cantilever truss detailed below.



3.6 STRUCTURES WITH INTERNAL HINGED CONNECTIONS

3.6.1 General Remarks

- Span of a structure is limited by its self-weight.
- There is a **span limit** beyond which a simply-supported structure would collapse under its own weight.
- To increase **overall span**, intermediate supports must be provided. Unfortunately this would render the support reactions statically indeterminate.
- Internal hinged connections may be used in conjunction with additional (intermediate) supports.

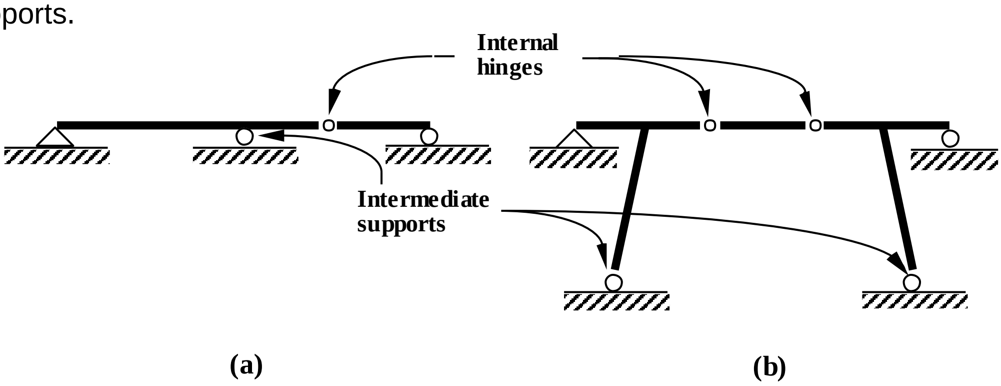


Fig. 3.20

3.6.2 Internal Hinges in Beams and Frames

- An internal hinge is represented by a circle at the joint of two members in bending.
- Structural characteristics of an internal hinge are such that it can carry or transfer forces, axial or otherwise but it is incapable of transmitting bending moment.
- At an internal hinge,

  $\Sigma M = 0$ . . . (3.20)

- One additional equilibrium equation for every internal hinge installed.

  R – I = 3   S.D.   . . . (3.21)

  R – I > 3   S.I.

  R – I < 3   Unstable (Always unstable, **necessary and sufficient conditions**)

  where R is the total number of reactions and I is the number of internal hinges.

  R – I < 3   (Always unstable, **necessary and sufficient conditions**)

  R – I ≥ 3   (May or may not be stable, **necessary but insufficient conditions**)
  (**Engineering inspection**:   Three hinges in a row?)

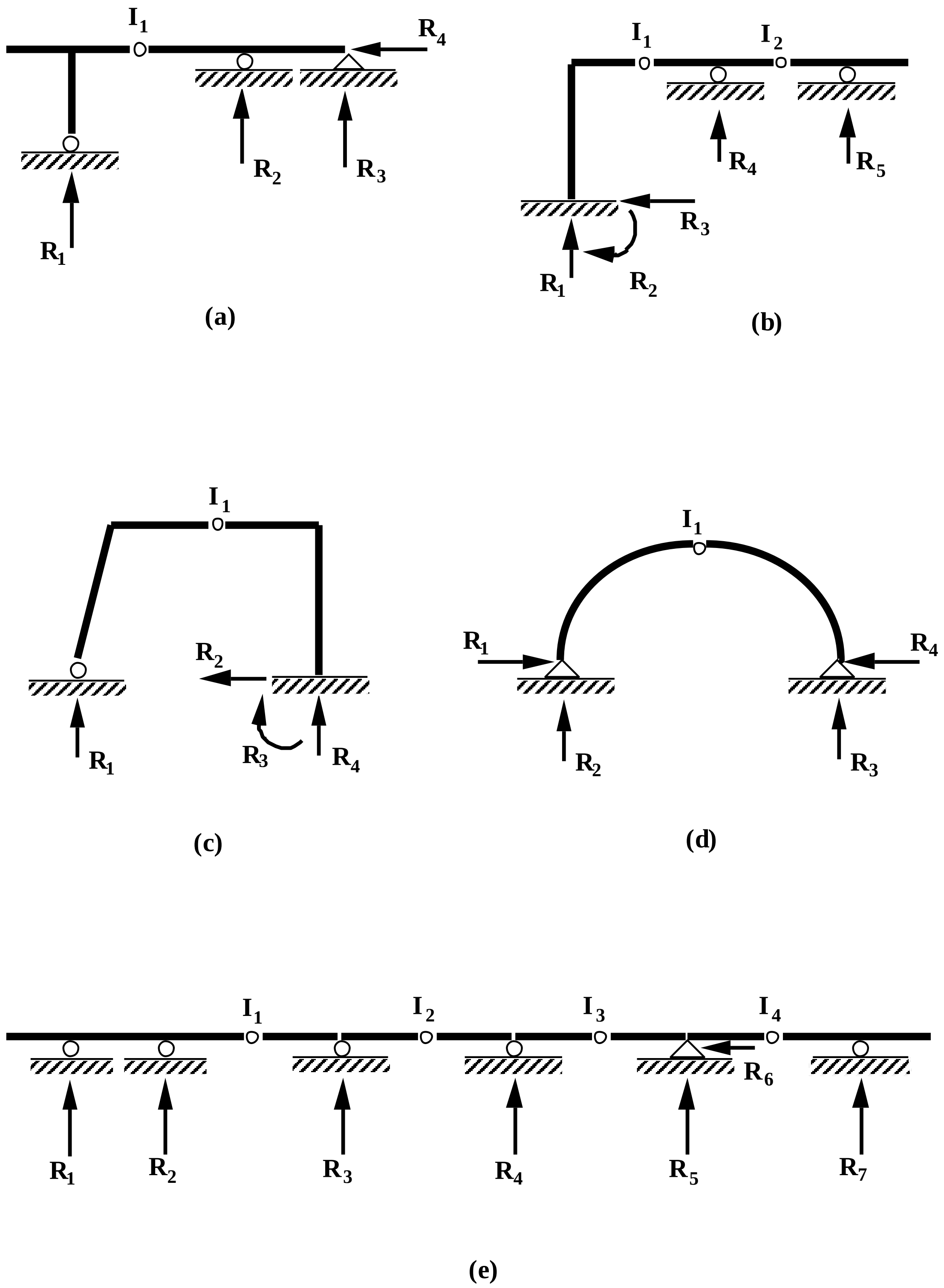


Fig. 3.21 Determinate structures

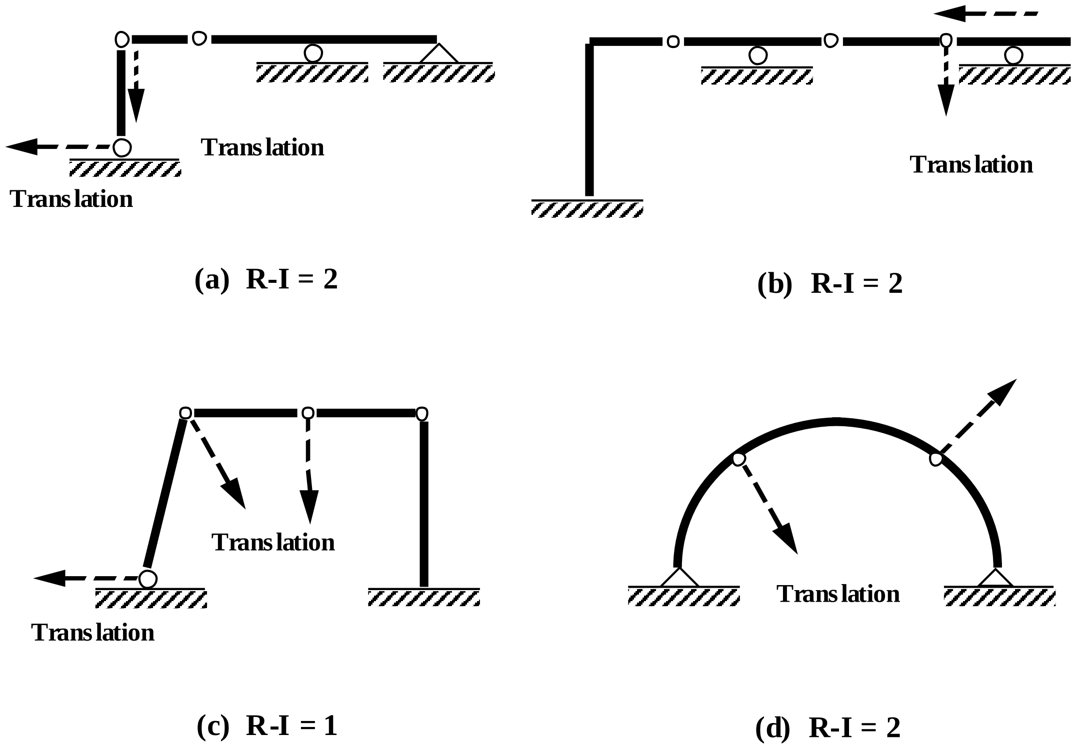


Fig. 3.22 Unstable systems

3.6.3 Analysis Procedure

Key points:

- Internal hinges can transmit forces (horizontal and vertical) but not moment.

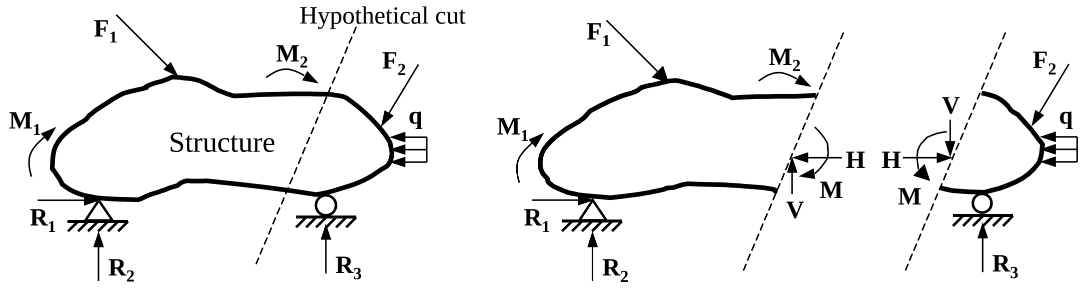
  $\Sigma F_X = 0$
  $\Sigma F_y = 0$
  $\Sigma M_{\text{about any point}} = 0$
  $\Sigma M_{@I} = 0$ (at every properly arranged internal hinge)

- Avoid solving simultaneous equations
- Free body concept

3.8 NOTES ON FREE BODY

3.8.1 Axiom

"*If the external actions (forces and moments) on a structure (Fig. 3.28(a)) are in equilibrium, then the actions (including internal ones) on any isolated portion of the structure are also in equilibrium*." Such an isolated portion is referred to as a **free body** (Fig. 3.28(b)).

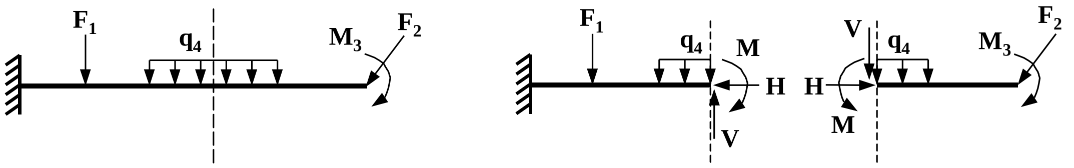


(a) Whole structure (b) Free bodies

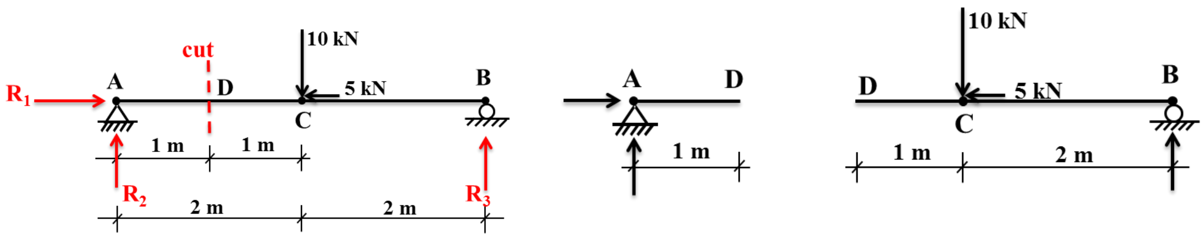
Fig. 3.28

Applications: Free body concept is important in obtaining solutions to many structural problems

3.8.2 Illustrative Examples

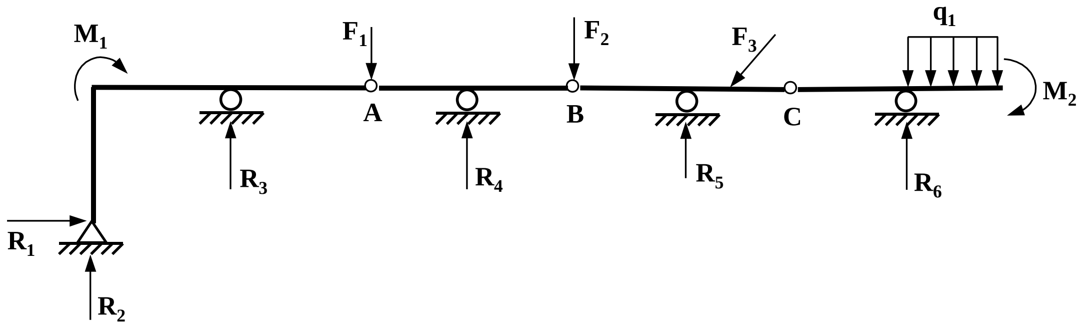


(a) Whole structure (b) Free bodies

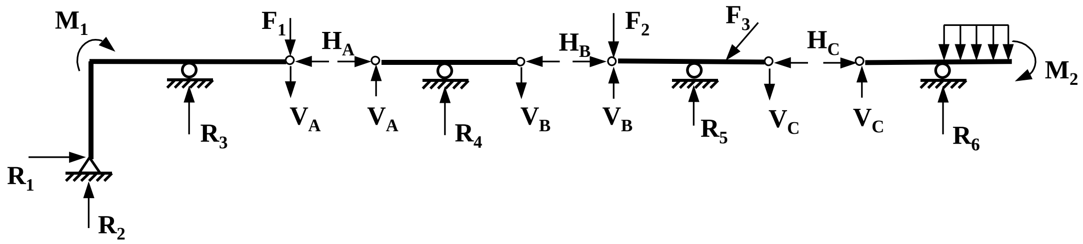


(c) Whole structure (d) Free bodies

Fig. 3.29



(a) Whole structure



(b) Free bodies

Fig. 3.30

Notes:

(1) The internal actions at the hypothetical "cut" of a free body are unknown. By appropriately applying the equilibrium equations (ΣFx=0, ΣFy=0, ΣM=0), they can be determined as well as the relevant reaction(s).

(2) The corresponding internal actions at the "cut" for two adjacent free bodies are equal in magnitudes but opposite in direction. Consequently, when the adjacent free bodies are "re-joined", the corresponding internal actions cancel each other out, thereby reverting to the "uncut" condition.

(3) External concentrated force at an internal hinge may be assigned to either of the two adjacent free bodies.

Analysis Procedure:

(1) Ensure that the given structure is stable and S.D. (R-I=3, R-I>3, R-I<3)

(2) Separate the structure into free bodies by hypothetically dismantling internal hinge(s) as appropriate

(3) For each free body, apply

- $\Sigma F_X = 0$, $\Sigma F_Y = 0$, $\Sigma M = 0$
- solve for unknown external reactions and internal forces (H & V) at internal hinge(s)
- It may be necessary to consider all free bodies including the original structure in order to compute all unknown reactions

(4) Perform a static check on whole structure

3.8.3 Example

Given a continuous structure as shown in Fig. 3.31, compute the reactions at supports A, B & D and the internal actions at C.

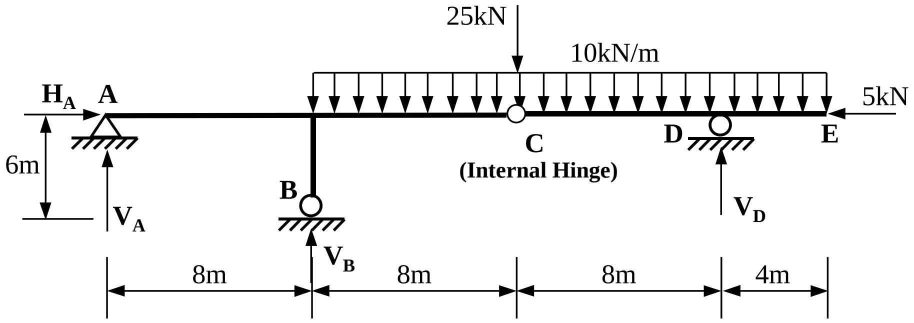


Fig. 3.31

Example 3.5 A three-hinged arch is shown in Fig. 3.23(a). Compute the reactions at supports A and C.

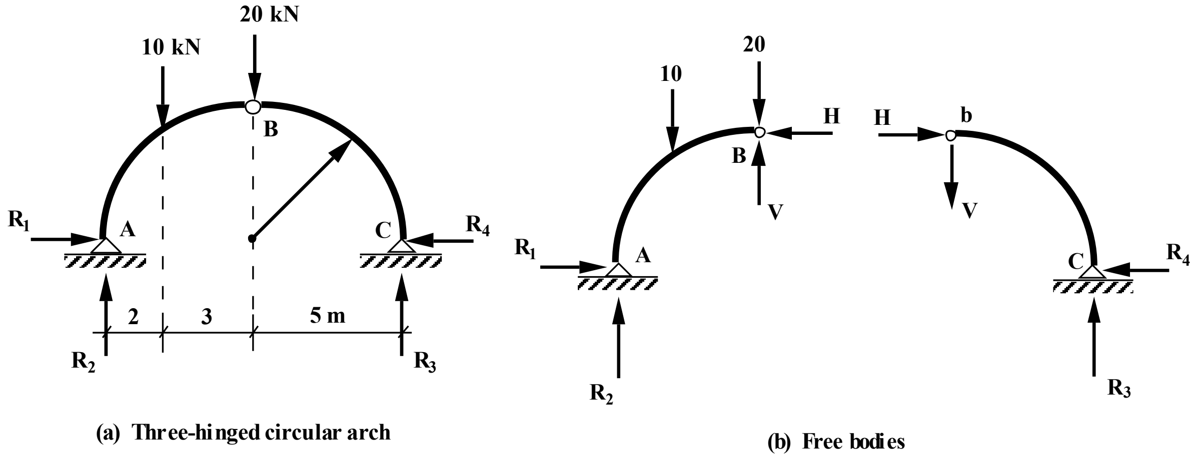
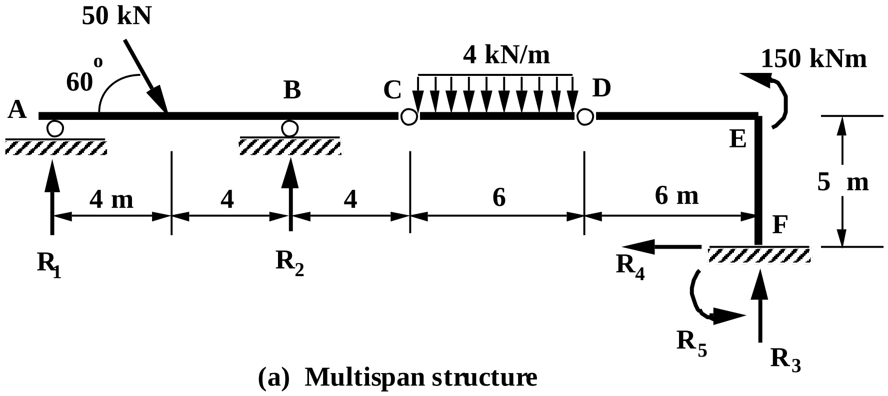


Fig. 3.23

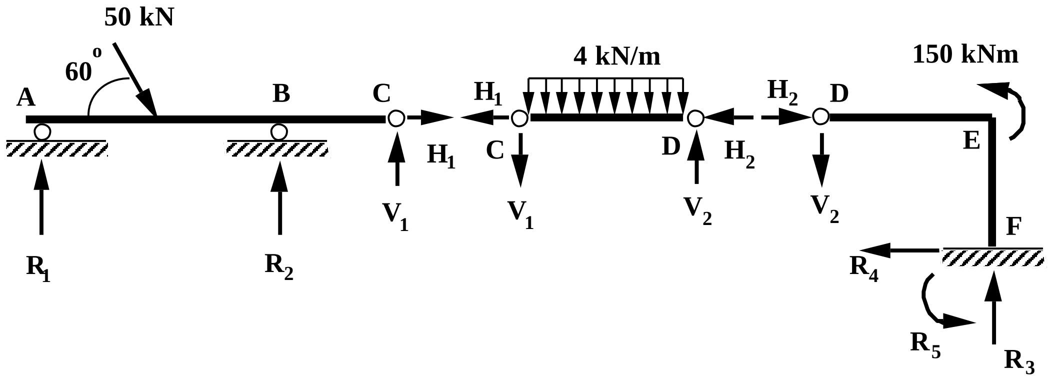
Step 1: Resolve the structure into two free bodies as shown in Fig. 3.23(b). The unknown internal forces are H and V.

Step 2: An inspection of the two free body diagrams indicates that there are four unknowns in each of them and none is solvable. This means that neither of these substructures should be analysed first. Now return to the original structure (Fig. 3.23(a)).

Example 3.6 For the multispan structure shown in Fig. 3.24(a) compute the reactions at all the supports.



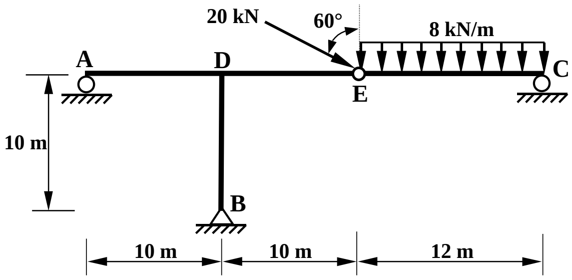
(a) Multispan structure



(b) Free body ABC (c) Free body CD (d) Free body DEF

Fig. 3.24

Example A pedestrian bridge is idealised below. For the given loading, compute the reactions at supports A, B and C, and indicate their directions.



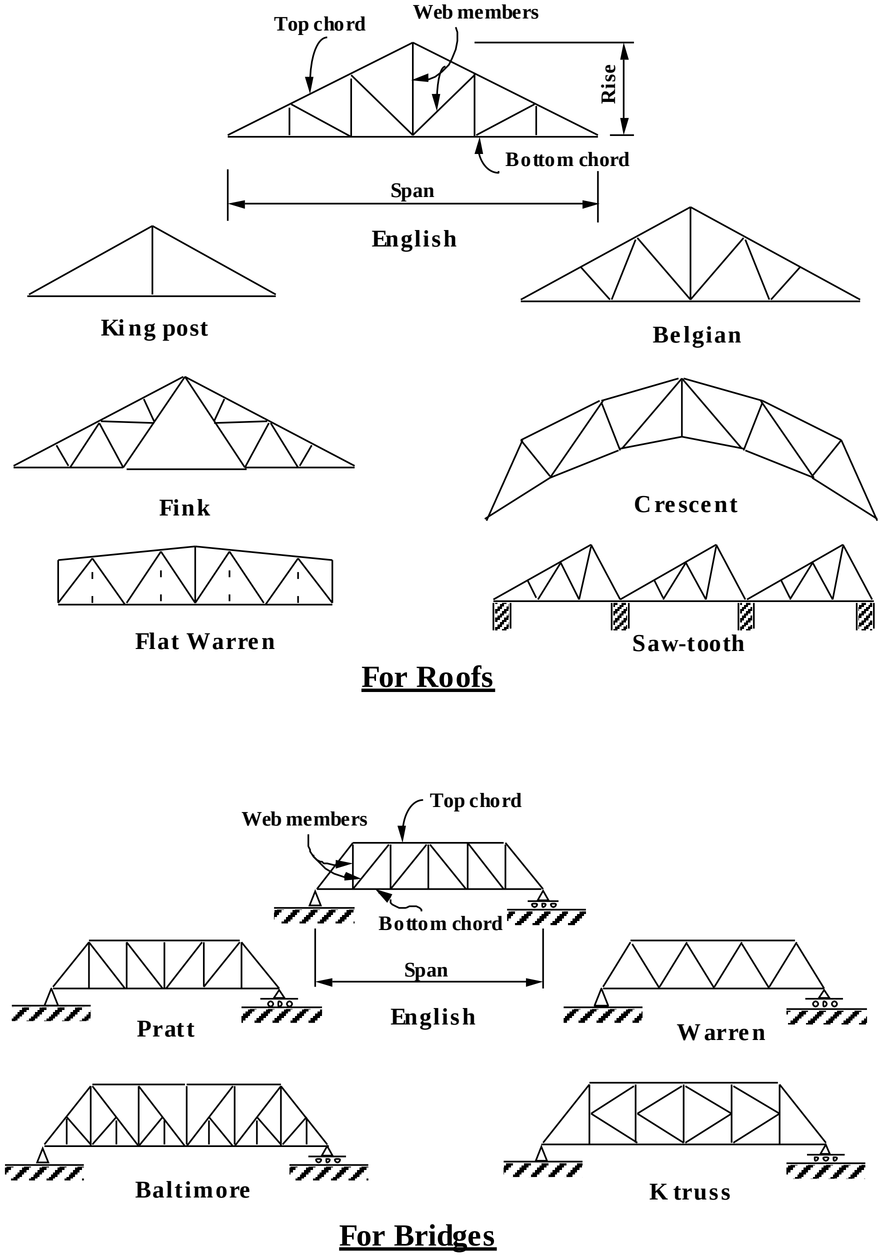
CHAPTER 4 ANALYSIS OF TRUSSES

OBJECTIVES AND EXPECTED OUTCOMES

- *Understand different types of trusses*
- *Distinguish statically determinate and indeterminate trusses*
- *Understand zero-force members*
- *Distinguish and apply appropriately method of joints and method of sections in truss analysis*

4.1 INTRODUCTION

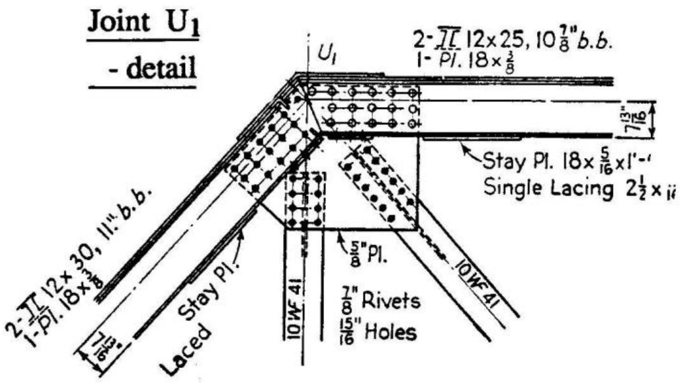
4.1.1 Types of trusses

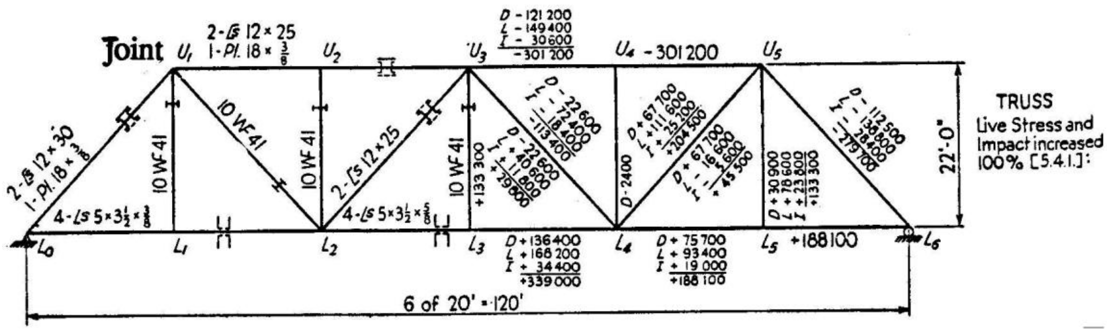


Typical Trusses

4.1.2 Construction and Nature of Joints

Real joints — rigid





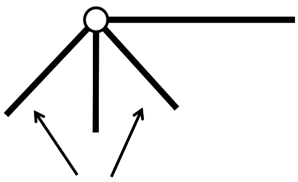
Assumptions:

(1) All joints are hinged (capable of transmitting two independent forces and at which there exists no bending moment). This is because a truss is by definition configured by the "**triangulation**" of two-force members.

(2) Load applied at joints

(3) Because of (1) & (2), all members are two-force members

(*due to* ***triangulation****, the predominant force in a truss member is an axial one, bending moment is secondary which maybe negligible at least in preliminary design considerations*)

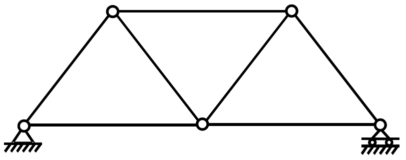


Two force members (only tension or compression (no moments))

4.2 STABILITY AND DETERMINACY

4.2.1 Criteria

Purpose of an analysis is to determine the nature and magnitude of each of the member forces



- Truss is assembly of two-force members. Hence total number of unknown forces (internal) = **m**
- Total number of unknown reactions (external) = **R**
- Two-force members and support reactions meet at hinged truss joints. Hence at every joint the member forces are "concurrent".

  $\Sigma F_X = 0$

  $\Sigma F_Y = 0$

  Total number of equilibrium eqs. = **2J** (i.e. 2 per joint)
  Total number of unknowns = **m+R**

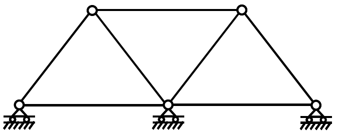
  Therefore

m + R = 2J	Statically Determinate (*)
m + R > 2J	Statically Indeterminate (*)
m + R < 2J	Unstable

  (*): Necessary but insufficient condition

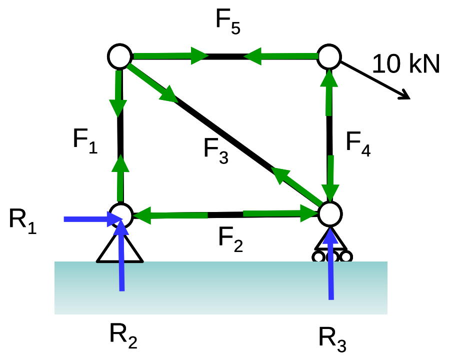
  Engineering inspection:

  (1) ensure support system is stable



  (2) ensure triangulation exists throughout (triangulation rule)

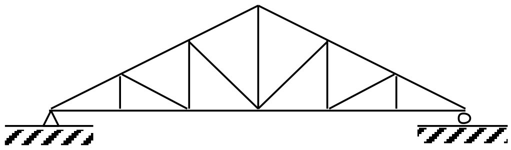
Simple example:



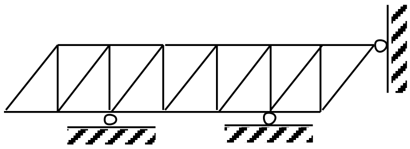
Total number of equilibrium equations:

Total number of unknowns:

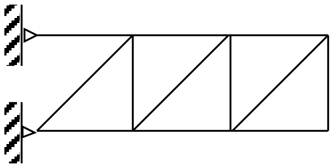
4.2.2 Illustrative Examples



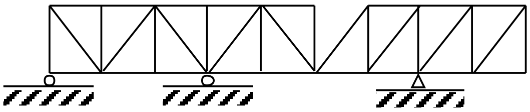
m=21; R=3; J=12
(a)



m=25; R=3; J=14
(b)

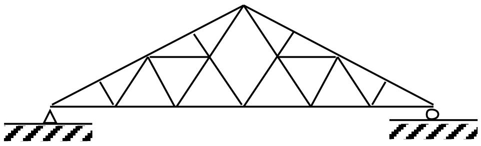


m=12; R=4; J=8
(c)

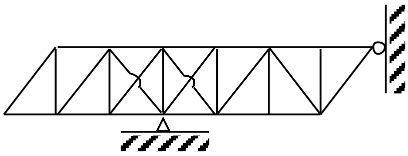


m=36; R=4; J=20
(d)

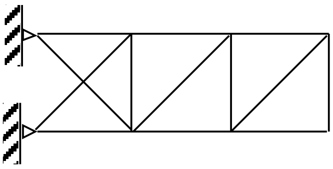
Statically determinate trusses



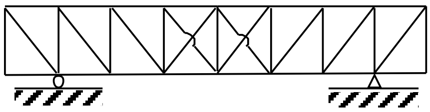
m=30; R=3; J=16
RM=1
(a)



m=27; R=3; J=14
RM=2
(b)



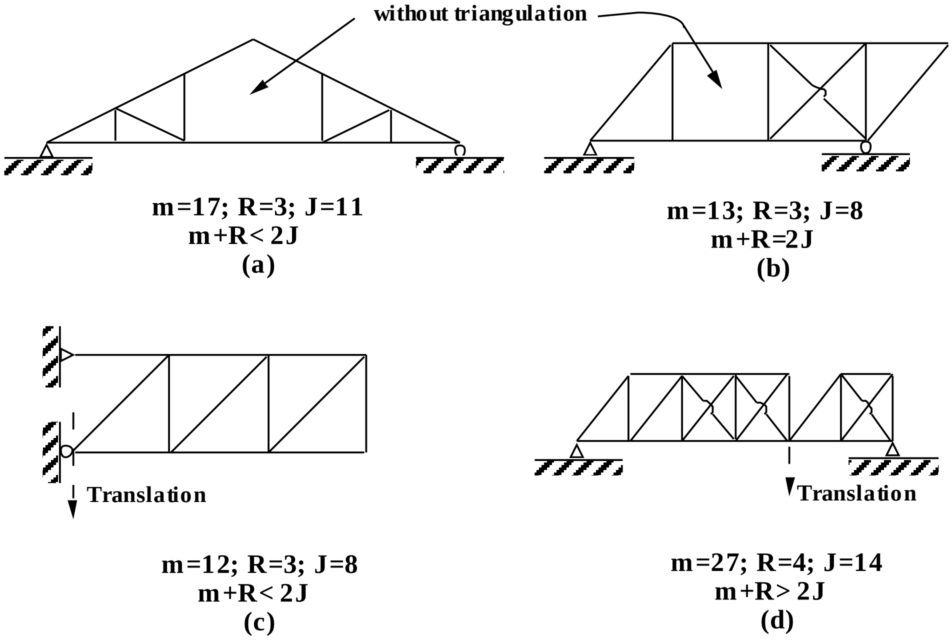
m=13; R=4; J=8
RM=1
(c)



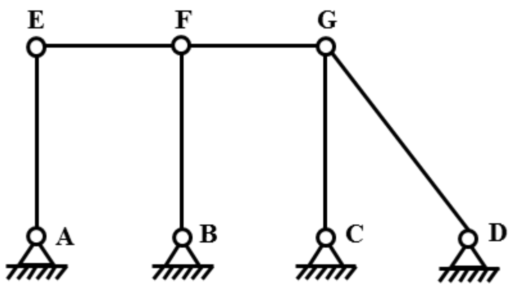
m=35; R=3; J=18
RM=2
(d)

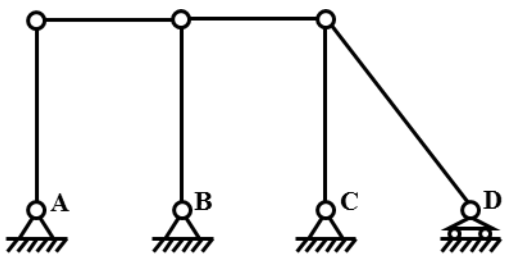
RM=Number of Redundant Members

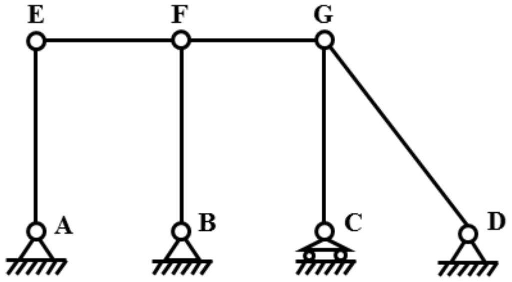
Statically indeterminate trusses



Unstable trusses

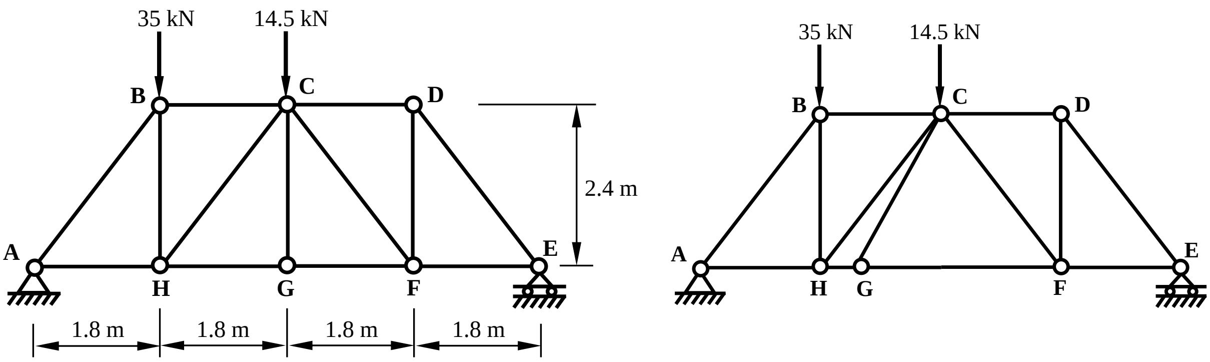




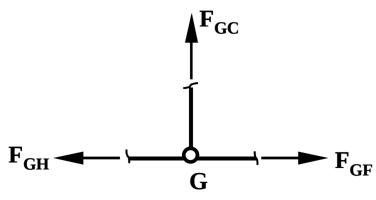


4.3 ZERO-FORCE MEMBERS

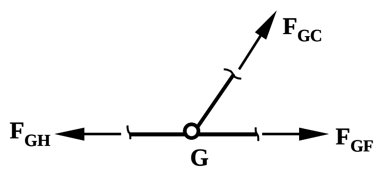
- Special case
- Zero-force member, or null member
- Null joint – the joint to which all null members are connected to
- simplifying solution



(a)

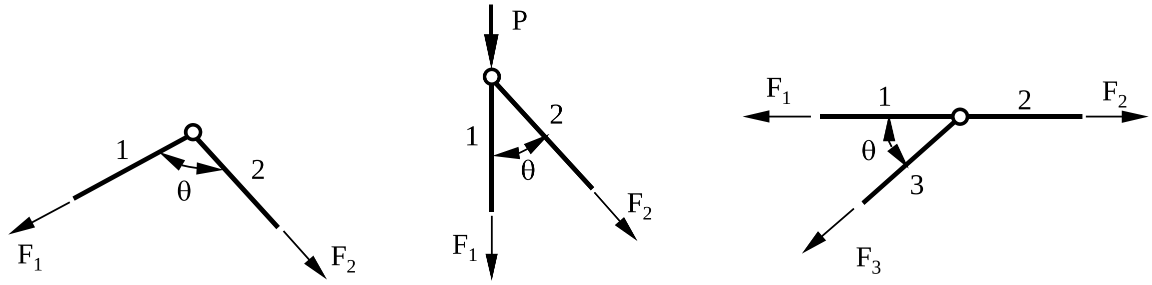


(b)



(c)

Three special cases:



(a) (b) (c)

(a) If there is no external load acting on the joint of two non-collinear members (1 and 2), both members are then zero force members regardless of the angle θ. Or,

$F_1 = F_2 = 0$

Prove: Take x along Member 1
$\sum F_y = 0$: $F_2 = 0$
$\sum F_x = 0$: $F_1 = 0$

(b) If an external load is applied at the joint of two non-collinear members (1 and 2), and it is collinear with one of the member forces (F_1 in this case), the other member is then a zero force member regardless of the angle θ. Or,

$F_2 = 0$
$F_1 = -P$

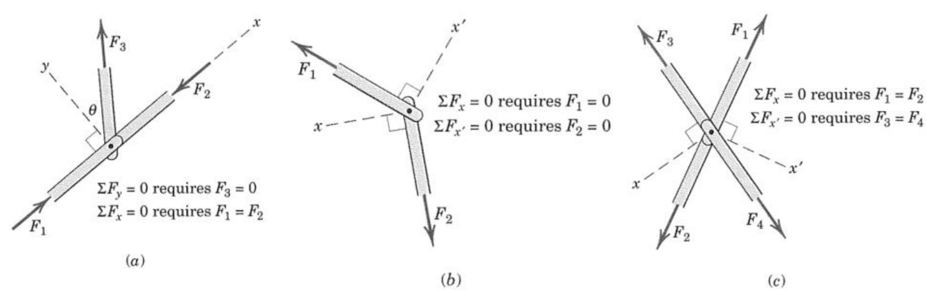
Prove: Take x along Member 1 and force P
$\sum F_y = 0$: $F_2 = 0$
$\sum F_x = 0$: $F_1 = -P$

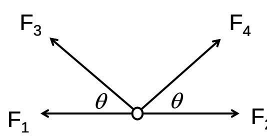
(c) For a three-member joint without external load acting on it, if two members are collinear (1 and 2), the third member (3 in this case) is a zero force member regardless of the angle θ. Or,

$F_3 = 0$
$F_1 = F_2$

Prove: Take x along Members 1 and 2
$\sum F_y = 0$: $F_3 = 0$
$\sum F_x = 0$: $F_1 = F_2$

Practical situation:





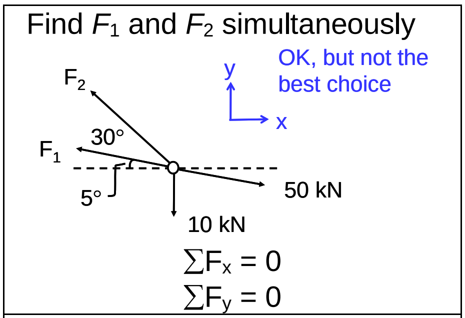
Even though F_3=0, the member is useful to prevent buckling

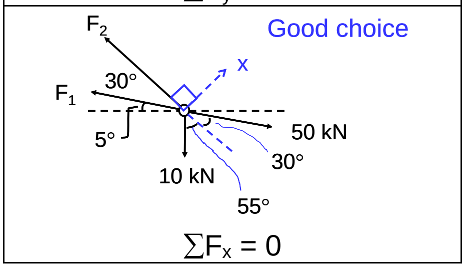
F_1=F_2=0, but if a load was placed on the joint this would change

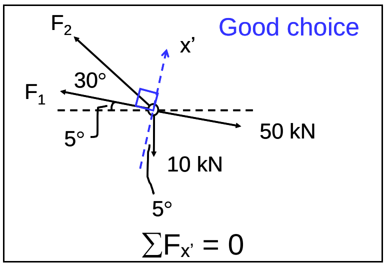
$\sum F_{y'} = 0$, F_1=F_2
$\sum F_y = 0$, F_3=F_4

$\sum F_y = 0$, $F_3 = -F_4$

Choice of coordinate axis:







4.4 METHODS OF ANALYSIS

Compute:

- reactions (external)
- member forces (internal axial force)

Methods:

- Method of joints
- Method of sections

4.4.1 Method of Joints

Features:

- Isolate each truss joint as a free body
- $\Sigma F_X = 0$; $\Sigma F_Y = 0$ (concurrent force system)
- Solve a maximum of two unknown forces per joint

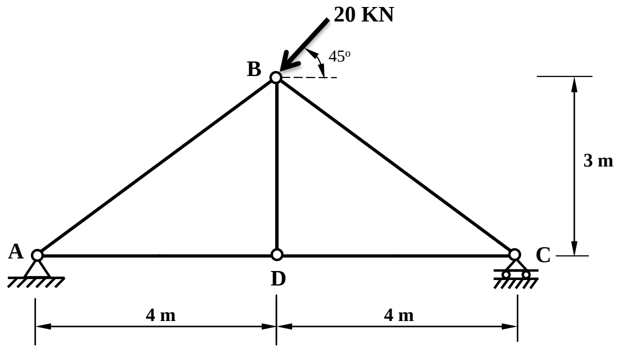
Applications:

- Complete analysis of a truss structure

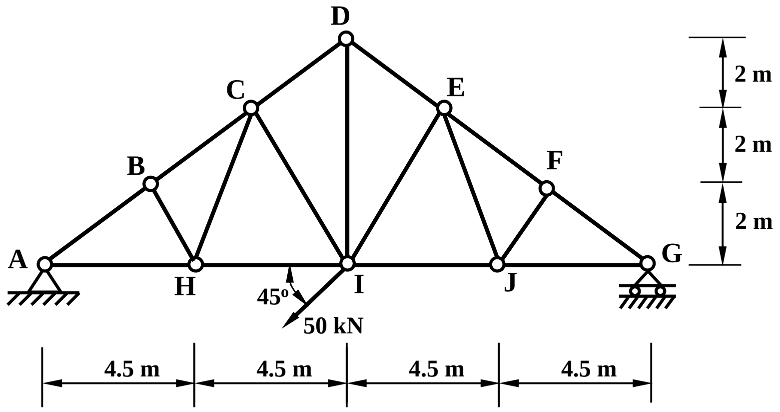
Analysis procedure:

(1) Compute the reactions. If necessary check first to ensure that the truss is statically determinate.
(2) Start with a joint where there are at most only two unknown forces.
(3) Solve the unknown force(s) by imposing $\Sigma F_X = 0$ and $\Sigma F_Y = 0$.
(4) Proceed to the next joint.
(5) Repeat steps (3) and (4) until all forces are determined.
(6) A static check would be available at the last joint to be considered.

Example 1 Determine the magnitude and nature of forces in each member of the truss. Use Method of joints.

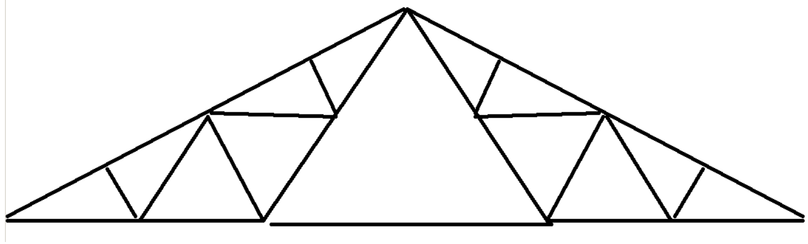


Example 2 Determine the magnitude and nature of all member forces for the Belgian truss. Use Method of joints.



4.4.2 Method of Sections

Fink Truss



Features:

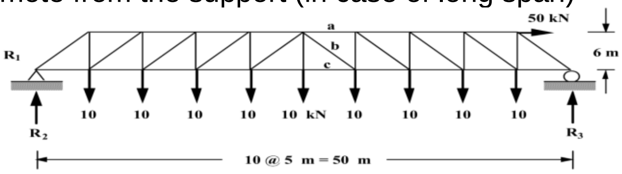
- Isolate selected parts of the truss as free bodies
- $\Sigma F_X = 0$; $\Sigma F_Y = 0$, $\Sigma M = 0$ (non-concurrent force system)
- Solve a maximum of three unknown forces at cut

Applications:

- When only a few member forces are to be determined
- When there is an impasse for the method of joints to analysis a truss completely (Fink truss)
- When concerning members remote from the support (in case of long span)

Find the forces in members *a, b, c*

Hard work by the Method of Joints!



Analysis procedure:

(1) Compute the reactions of the appropriate support(s). If necessary, also check first to ensure that the truss is statically determinate.

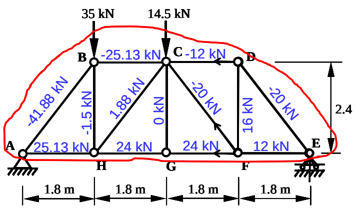
(2) Isolate a portion of the truss as a free body where there are at most three unknown member forces.

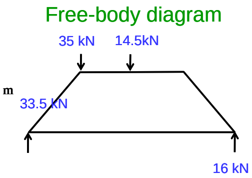
(3) Impose the three static equations to compute the unknown forces.

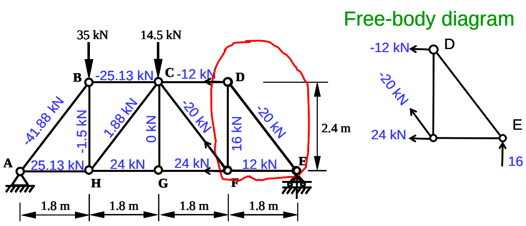
$\Sigma F_X = 0$; $\Sigma F_Y = 0$, $\Sigma M = 0$

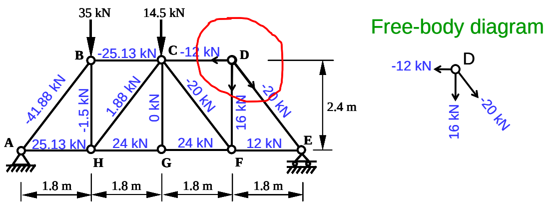
(4) Repeat steps (2) and (3) as necessary until all required results are obtained.

Any and every part of the structure is in static equilibrium

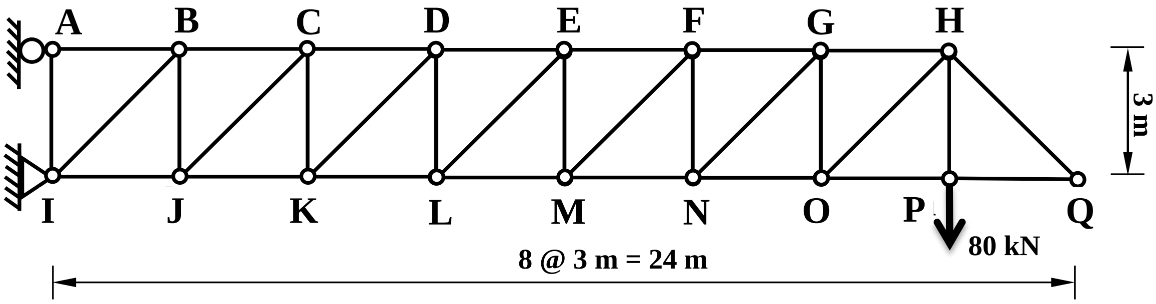




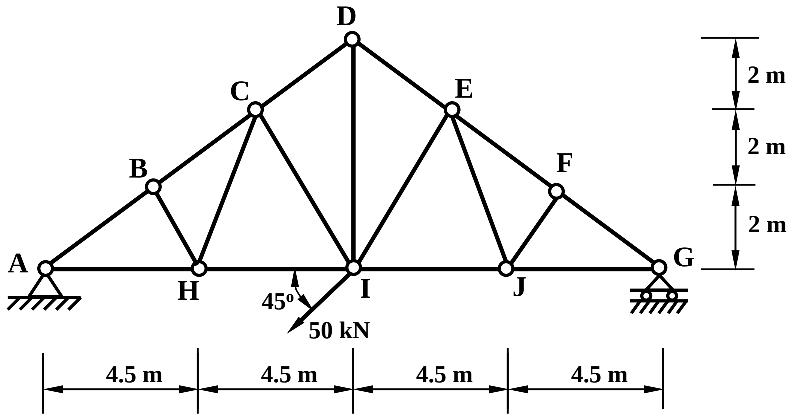




Example 3 Determine the magnitude and nature of the axial forces in members 1 (CD), 2 (KD), 3 (KL) and 4 (FN). Use Method of section.



Example 4 Determine the magnitude and nature of the axial forces in members 1 (IJ), 2 (IE) and 3 (DE). Use Method of section. (same as Example 2)



Summary:

(1) S.D., S.I., stable, unstable?
(2) Identify null members
(3) Determine support reactions
(4) Method of joints or Method of sections?

Method of joints:

- Start at a joint with only two unknown forces
- Align coordinate axis so it is 90° to one unknown
- Move joint by joint around the truss
- Moment equation is only used for finding external reactions if needed

Method of sections:

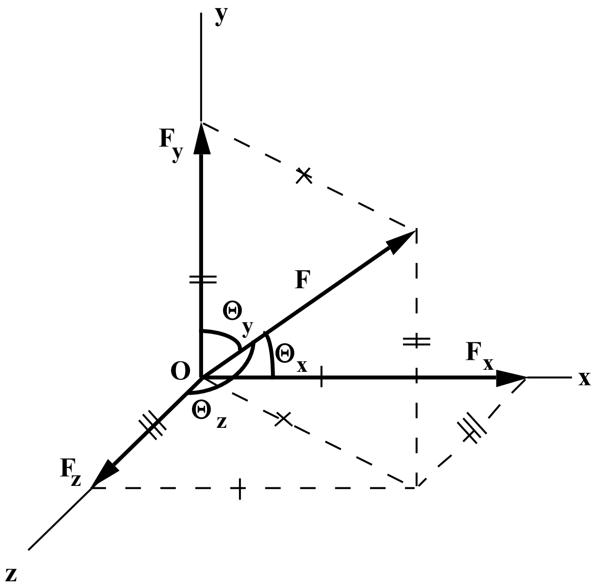
- Useful 'short-cut' to finding forces in particular members
- Uses all three equations of equilibrium
- Solves a maximum of 3 unknown forces at a cut

CHAPTER 5 ANALYSIS OF FORCES AND MOMENTS IN SPACE

OBJECTIVES AND EXPECTED OUTCOMES

- *Apply equilibrium equations to 'simple' 3D structures in order to determine reaction forces and moments*

5.1 COMPONENTS OF A FORCE IN SPACE



Force F has an inclination Θ_x w.r.t. the x - axis, Θ_y w.r.t. the y-axis and Θ_z w.r.t. the z-axis. The magnitudes of the three components,

$$F_x = F \cos \Theta_x = F \times a/r$$

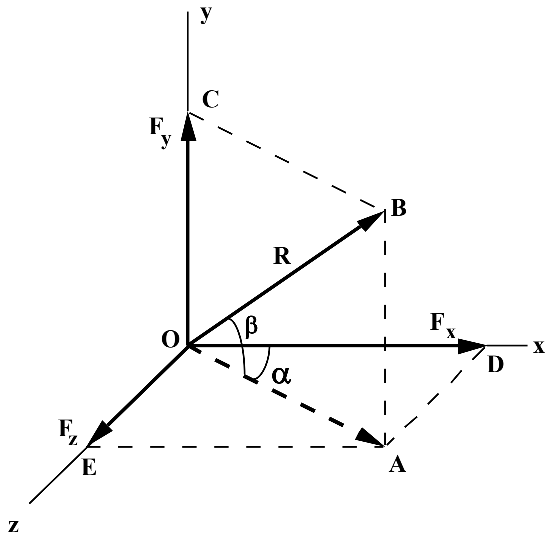
$$Fy = F \cos \Theta_y = F \times b/r$$

and $$F_z = F \cos \Theta_z = F \times c/r$$

RESULTANT FORCE

Concepts

- Force F may be seen as the resultant force, or simply the resultant, of its components.
- 2 components F_2 and F_3 → resultant F_1 (law of sine or law of cosine).
- More than 2 components → resultant F_1 (reversed step-by-step procedure).



- Magnitude of resultant R

$$R^2 = OA^2 + AB^2 = OE^2 + OD^2 + OC^2$$

i.e. $$R^2 = F_x^2 + F_y^2 + F_z^2$$

Or, $$R = \sqrt{F_x{}^2 + F_y{}^2 + F_z{}^2}$$

- Direction of R is expressed by the inclinations α and β

$$\alpha = \tan^{-1}\frac{AD}{OD} = \tan^{-1}\frac{F_z}{F_x}$$

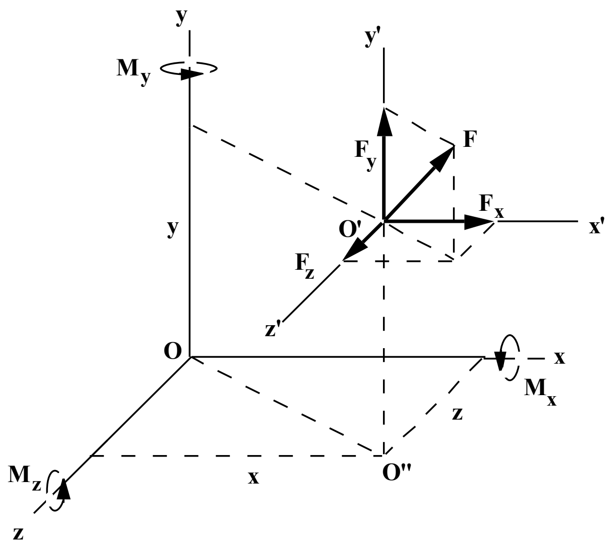
and

$$\beta = \tan^{-1}\frac{AB}{OA} = \tan^{-1}\frac{F_y}{\sqrt{F_x{}^2 + F_z{}^2}}$$

Comments

- The line of action of the resultant of a concurrent force system always passes through the concurrent point.
- As nonconcurrent forces do not have a concurrent point, the location of the resultant is also unknown in addition to its direction and magnitude.

MOMENTS PRODUCED BY A FORCE IN SPACE



- F acting at O' (x, y, z) does not pass through the origin O
- F produces moments M_x, M_y and M_z respectively about the x, y and z axes
- "right-hand-screw rule" is used

Moment about the x-, y- and z-axes:

$$M_x = -F_y \cdot z + F_z \cdot y + F_x\,(o) = -F_y \cdot z + F_z \cdot y$$

$$M_y = -F_z \cdot x + F_x \cdot z$$

and

$$M_z = -F_x \cdot y + F_y \cdot x$$

Note that the above equations can be "generated" by "rotating" the subscripts for M and F and the coordinates sequentially, i.e. x to y, y to z and z to x.

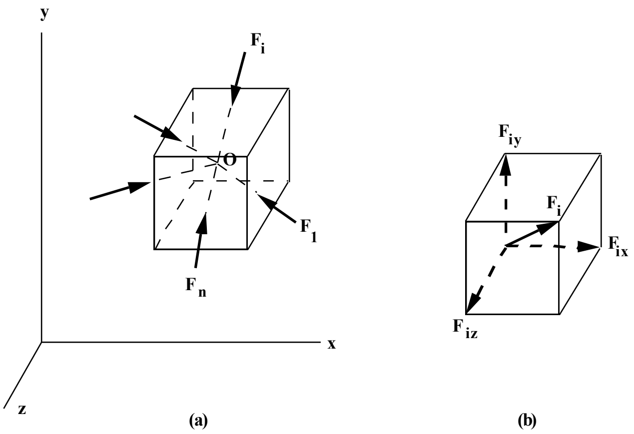
General rule:

A force does not produce a moment about an axis if its line of action

- is parallel to that axis
- passes through that axis (d = 0)

EQUILIBRIUM IN 3-D SPACE

Concurrent Forces



(a) (b)

$$F_{1x} + F_{2x} + F_{3x} + \ldots + F_{ix} + \ldots + F_{nx} = 0$$
$$F_{1y} + F_{2y} + F_{3y} + \ldots + F_{iy} + \ldots + F_{ny} = 0$$
$$F_{1z} + F_{2z} + F_{3z} + \ldots + F_{iz} + \ldots + F_{nz} = 0$$

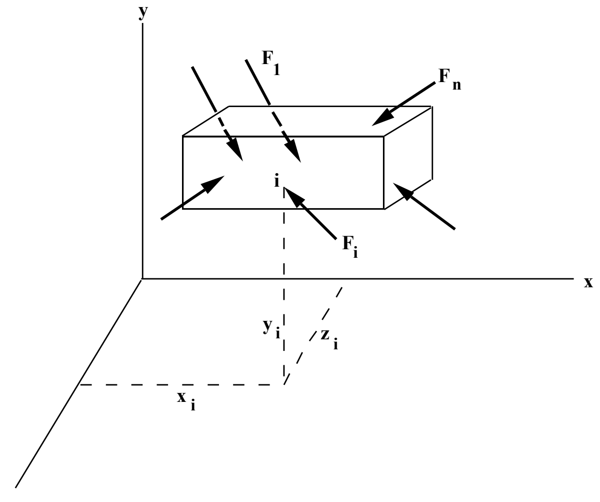
Another way to write this is:

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_z = 0$$

3 eqs. must be satisfied simultaneously

Case of **concurrent forces** – don't need to worry about moments because all forces pass through the same point ($\Sigma M_x = \Sigma M_y = \Sigma M_z = 0$)

Non-concurrent Forces



(May also include some applied moments)

$$\sum F_x = 0 \;,\; \sum F_y = 0 \;,\; \sum F_z = 0$$

Moments about x-axis:

$$\sum_{i=1}^{n} \left(F_{iz} y_i + F_{iy} z_i \right) + \sum_{j=1}^{m} M_{jx} = 0 \qquad \left(\sum M_x = 0 \right)$$

Moments about y-axis:

$$\sum_{i=1}^{n} \left(F_{ix} z_i + F_{iz} x_i \right) + \sum_{j=1}^{m} M_{jy} = 0 \qquad \left(\sum M_y = 0 \right)$$

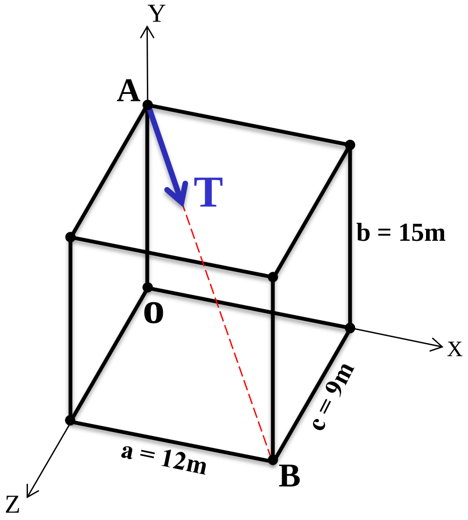
Moments about z-axis:

$$\sum_{i=1}^{n} \left(F_{iy} x_i + F_{ix} y_i \right) + \sum_{j=1}^{m} M_{jz} = 0 \qquad \left(\sum M_z = 0 \right)$$

(Some F_{ix}, F_{iy} and F_{iz} have negative values)

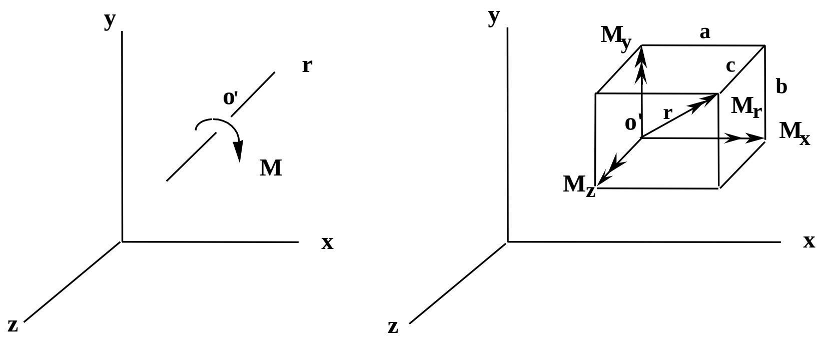
6 eqs. must be satisfied simultaneously

Example: Determine F_1, F_2, F_3, M_1, M_2, M_3 of the following non-current force system.



5.2 COMPONENTS OF A MOMENT

5.2.1 Concept

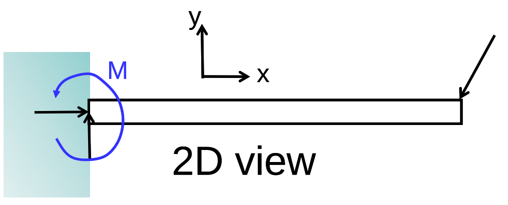


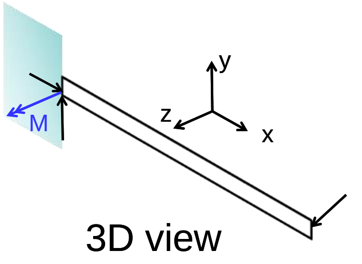
(a) (b)

- A moment about an axis r in space may be represented by a double-headed arrow via the right-hand screw rule.
- The arrow is a vector having the direction as shown and the magnitude equal to the moment M.
- The line of action of M coincides with axis r. Thereafter, the moment may be treated as if it is a force in space.

$$M_x = M \times a/r \qquad \text{. . . (5.7)}$$
$$M_y = M \times b/r \qquad \text{. . . (5.8)}$$
$$M_z = M \times c/r \qquad \text{. . . (5.9)}$$
$$r = \sqrt{a^2 + b^2 + c^2} \qquad \text{. . . (5.10)}$$

For the 2D problems that we considered so far, moments always acted in the z-direction

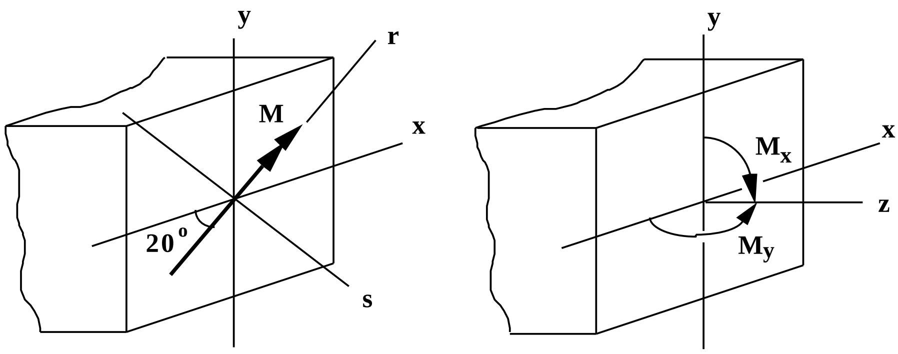




5.2.2 Numerical Example

- Due to symmetry of most structural designs, chances of an engineer having to deal with a moment acting about axes other than those parallel to the x, y or z axis are low.
- However in asymmetrical bending of beams, the moments act about an axis which is neither x nor y.

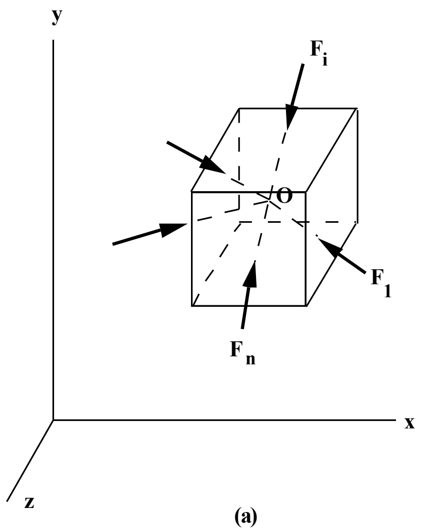
Example The figure shows an asymmetrically loaded beam. For the internal moment M = 400 kNm acting about axis r, compute the component moments about the x and y axes.



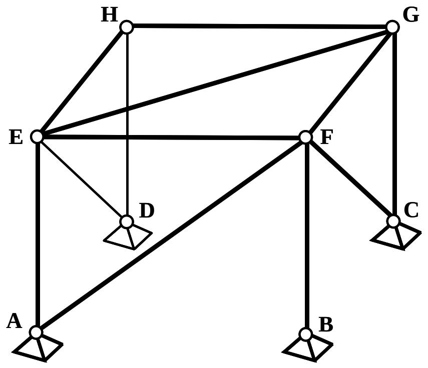
(a) (b)

Solution:

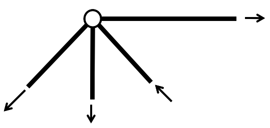
5.3 CONCURRENT FORCES



(a)

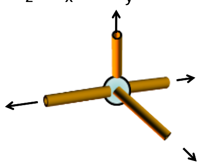


Model: pin joint connection & 2-force members (cannot transfer moments)



2D model for plane truss joint

Model: 'ball and socket' joint connection & 2-force members (cannot transfer M_z, M_x or M_y moments)



3D model for space truss joint

- Encountered mainly in space-truss typed structures.
- For each set of concurrent forces to be in equilibrium:

$$\Sigma F_X = 0$$
$$\Sigma F_Y = 0$$
$$\Sigma F_Z = 0$$

- Maximum number of unknown forces that can be analysed is three per concurrent point at any one time. *But 3 simultaneous equations is hard work by hand! (Use computer)*

5.3.1 Method of Projections

- We can solve for the forces in a space truss using a similar method to the method of joints for plane trusses.
- To reduce the number of simultaneous equations (3 unknowns) at each joint, project the concurrent forces onto a plane to eliminate some unknowns.
- In a plane only two unknowns can be solved at one time.
- Solution of the component forces in these planes leads to the solution of the 3-D unknown forces, via

$$F = F_x\ r/a = F_y\ r/b = F_z\ r/c$$

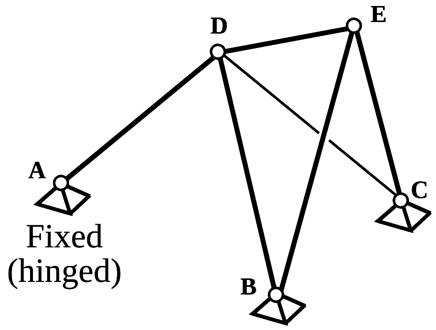
5.3.2 Determinacy and Stability of Space Trusses

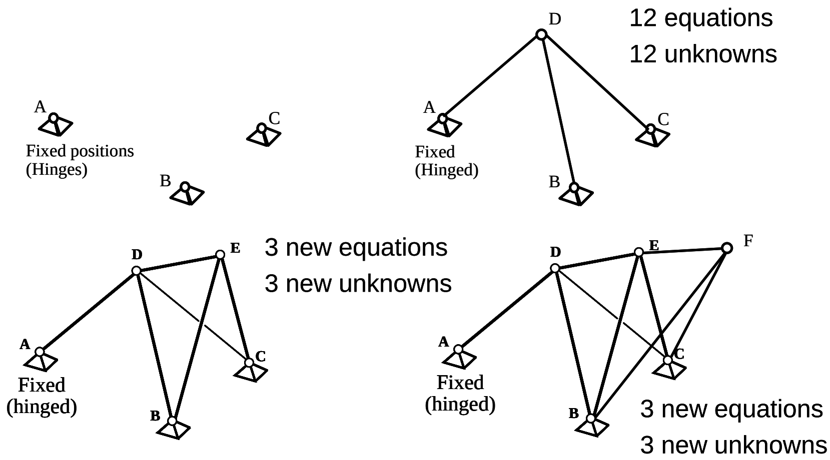
$m + R = 3J$	Statically Determinate (*)	. . . (5.11)
$m + R > 3J$	Statically Indeterminate (*)	. . . (5.12)
$m + R < 3J$	Unstable	. . . (5.13)

(*): Necessary but insufficient condition

m - number of members in the truss
R - number of unknown support reactions
J - number of truss joints

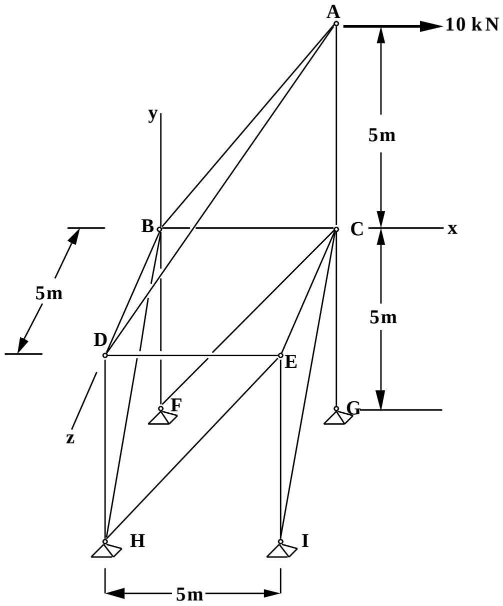
Space triangulation requirements ensure stability: with three "fixed" points, the fourth is fixed or stable.





"The truss is stable if the configuration is attained through a pyramid building process".

Tetrahedral pyramid building process

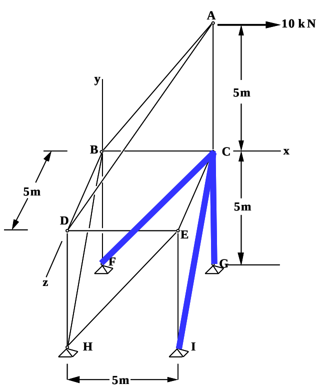


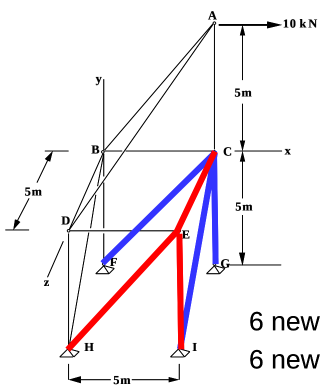
12 reactions

$$m + R = 3J$$

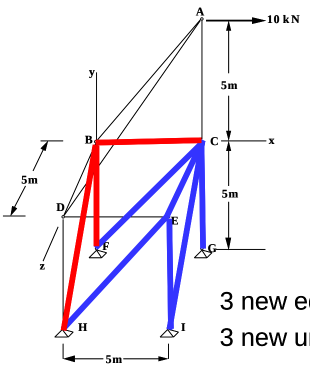
15 members

9 joints

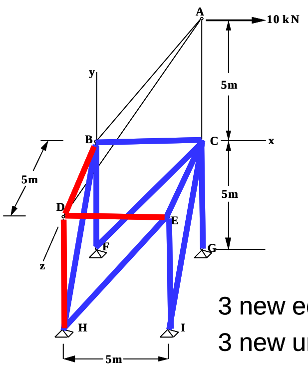




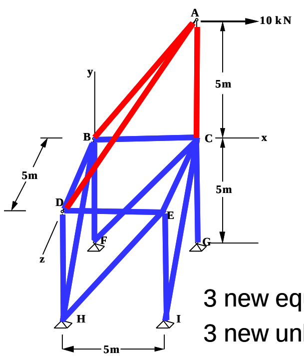
6 new equations
6 new unknowns



3 new equations
3 new unknowns



3 new equations
3 new unknowns



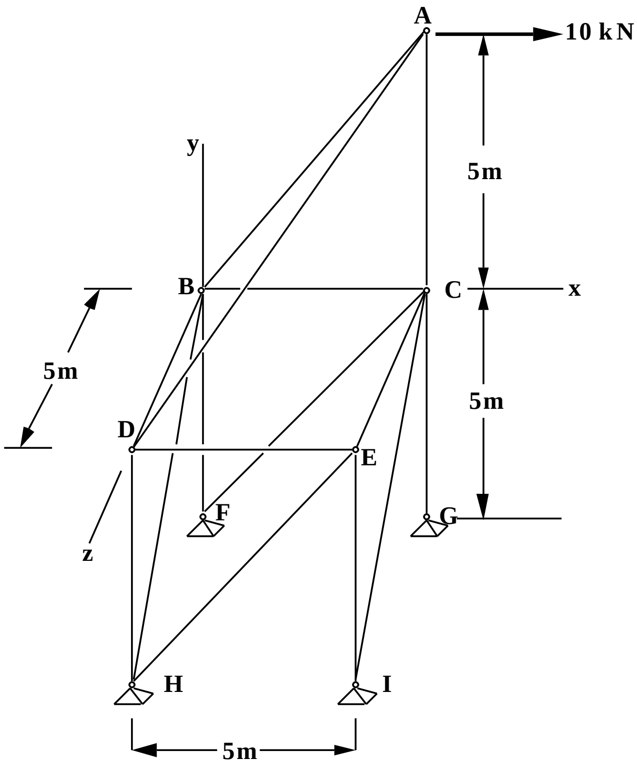
3 new equations
3 new unknowns

Null members and null joints in space truss

- For a space truss joint where 3 members meet, if there is no externally applied force, then all the members are null members and the joint is a null joint.
  (Use method of projection to prove)

- For a space truss joint, any one force out of the plane under consideration (i.e. into or out of the plane) must be equal to zero.

- **Or, if all forces act in one plane, the force out of that plane = 0**

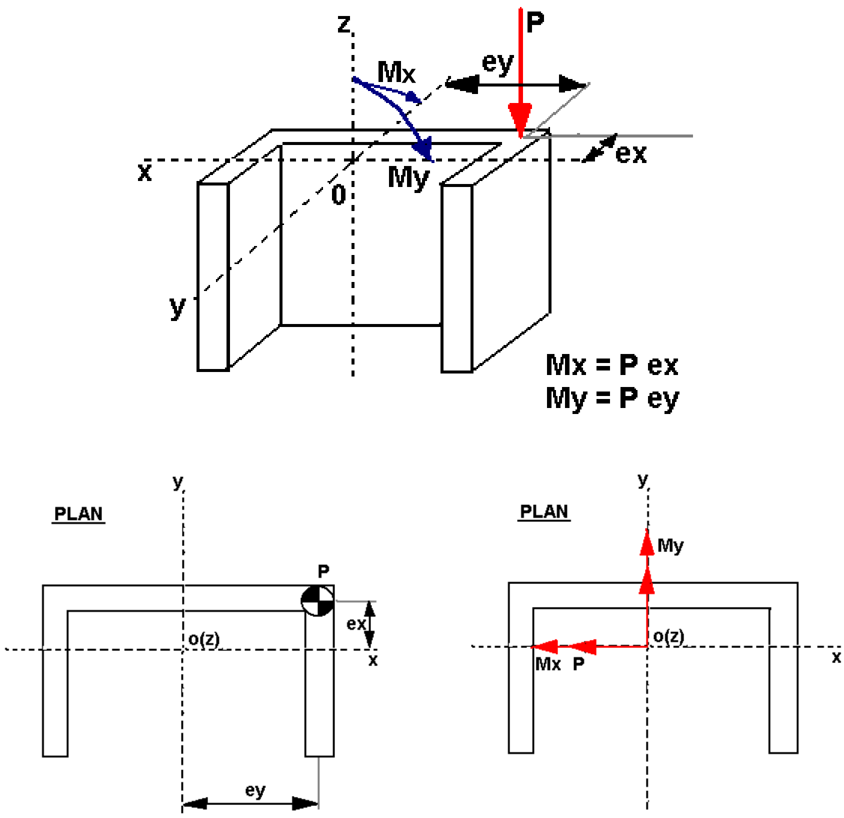


Identify null members and null joints

5.4 NONCONCURRENT FORCES

Encountered in space frames and other 3-D systems such as beams and columns subjected to eccentric loads and cantilever structures with back-stays

5.4.1 Moments about Structural Member Axes



In the design, it is necessary to transfer P to the point O, making it a concentric or a true axial load. Such a transfer must be compensated by two moments. They are,

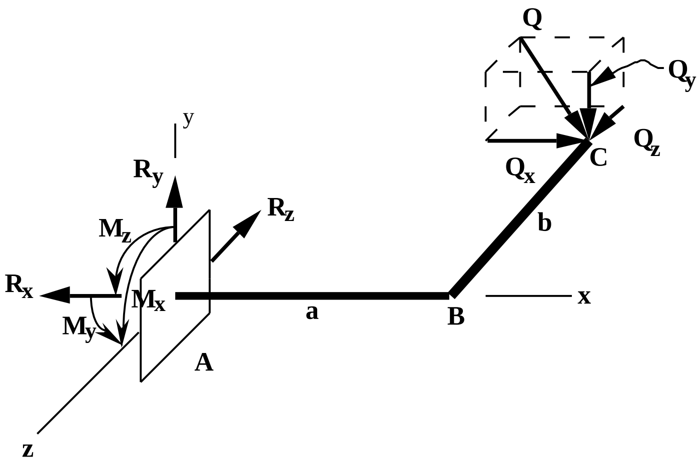
$$M_x = P \cdot e_x \qquad \text{. . . (5.14)}$$

and $$M_y = P \cdot e_y \qquad \text{. . . (5.15)}$$

5.4.2 Reactions of Cantilever Structures

As the built-in support can provide all the six force and moment reactions, a cantilever beam/column in space is stable and statically determinate.

A cantilever bent ABC subjected to a force Q in space at the tip C is detailed in figure.



The reactions at support A can be computed using Eqs. 5.1 to 5.6. Thus,

$\Sigma F_x = 0$:	$R_x = Q_x$
$\Sigma F_y = 0$:	$R_y = Q_y$
$\Sigma F_z = 0$:	$R_z = Q_z$
$\Sigma M_x = 0$:	$M_x = Q_y \cdot b$
$\Sigma M_y = 0$:	$M_y = Q_x \cdot b + Q_z \cdot a$
$\Sigma M_z = 0$:	$M_z = Q_y \cdot a$

The directions of the six reactions are as shown in figure.

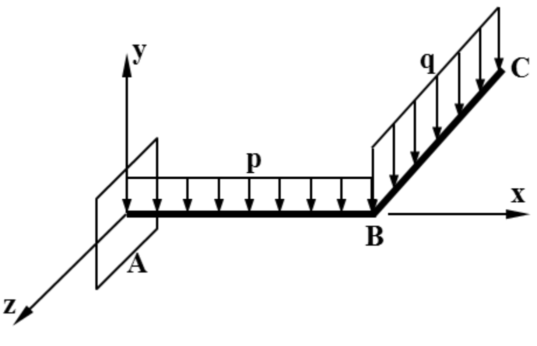


Figure below shows a bent beam subjected to 3 moment components M_{CX}, M_{Cy} and M_{CZ} applied at the tip C.

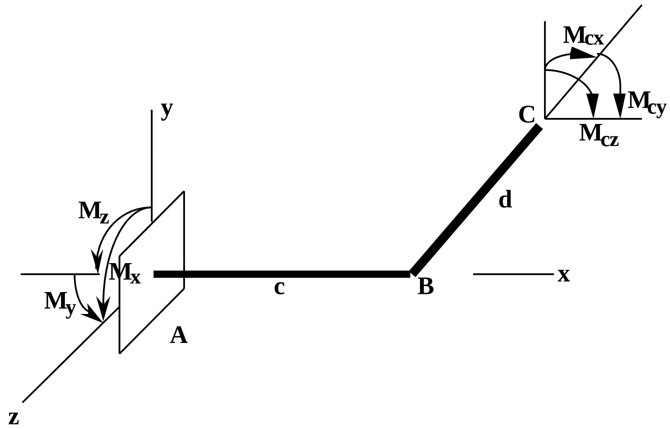


Fig. 5.10

Again Eqs. 5.1 to 5.6 may be applied to obtain all the six reactions at A. $\Sigma F_X = 0$: $R_X = 0$ $\Sigma F_y = 0$: $R_y = 0$ $\Sigma F_Z = 0$: $R_Z = 0$ $\Sigma M_X = 0$: $M_X = M_{CX}$ $\Sigma M_y = 0$: $M_y = M_{Cy}$ $\Sigma M_Z = 0$: $M_Z = M_{CZ}$	

- Note that the directions of the reaction moments are opposite to the applied moments.
- Also the answer would be the same if the moments were applied at B instead of C.
- While a load in space can produce both reaction forces and moments, a moment in space produces no reaction forces but only reaction moments at the cantilever support.

CHAPTER 6 SHEAR FORCE AND BENDING MOMENT IN BEAMS

OBJECTIVES AND EXPECTED OUTCOMES

- *Understand the internal actions of shear force and bending moment*
- *Understand the relationships between loading, shear force and bending moment*
- *Draw shear force diagrams and bending moment diagrams*

6.1 INTRODUCTION

- Beams are the most important class of elements in engineering structures. → Very important to study the shear force and bending moment developed in a beam as well as their distributions along the span of the beam.

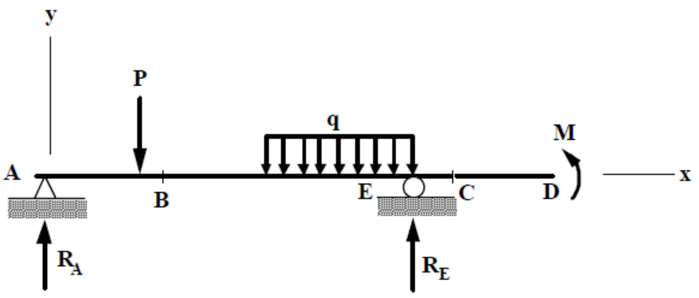
- Regardless of the construction material used (timber, concrete, steel, etc) the shear force and bending moment developed in the beam are computed the same way.

Symbols: S: Shear force
M: Bending moment

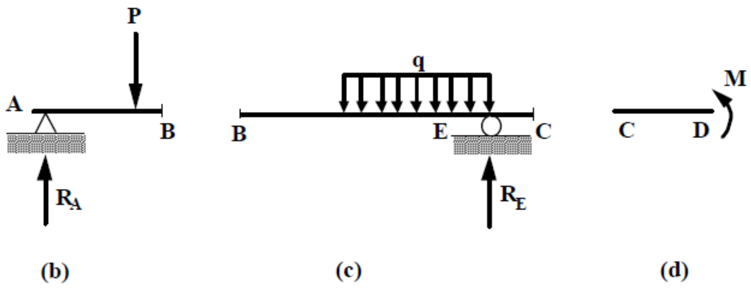
Units: S: kN or N
M: kNm or Nmm

Applications: S and M values are required in the design calculations for beams and other structural members subjected to bending, such as frame members.

Concepts: S and M are internal actions of beams. They are always referred to a given section (or point) of the beam concerned.



Free bodies must also be in equilibrium:



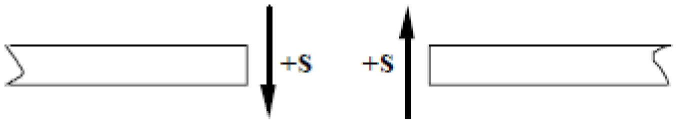
(i) The shear forces acting on the two cross sections at either side of a cut are equal in magnitudes and opposite in directions.

(ii) The bending moments acting on the two cross sections at either side of a cut are equal in magnitudes and opposite in directions.

Sign Conventions:

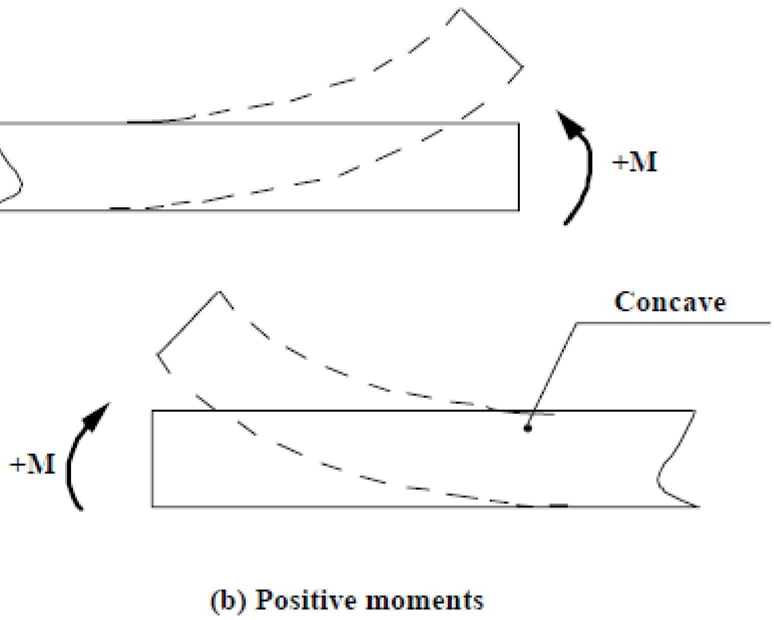
(i) A positive shear is one that tends to produce a clockwise rotation of the free body on which it acts.

(ii) A positive bending moment is one that tends to bend the free body into a concave shape.



(a) Positive shears

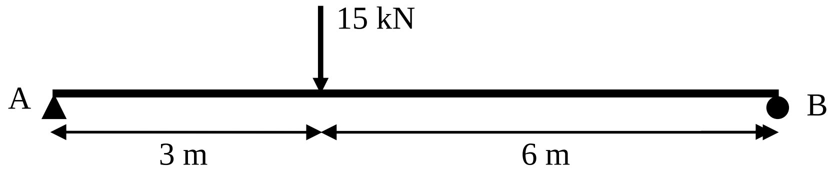
(causing clockwise rotation of freebody on which it acts)



(b) Positive moments

(bending the beam into a concave shape)

Example 1: Obtain the values of S and M at the sections where marked for the beam as shown.



Solution:

Note: **S**

M

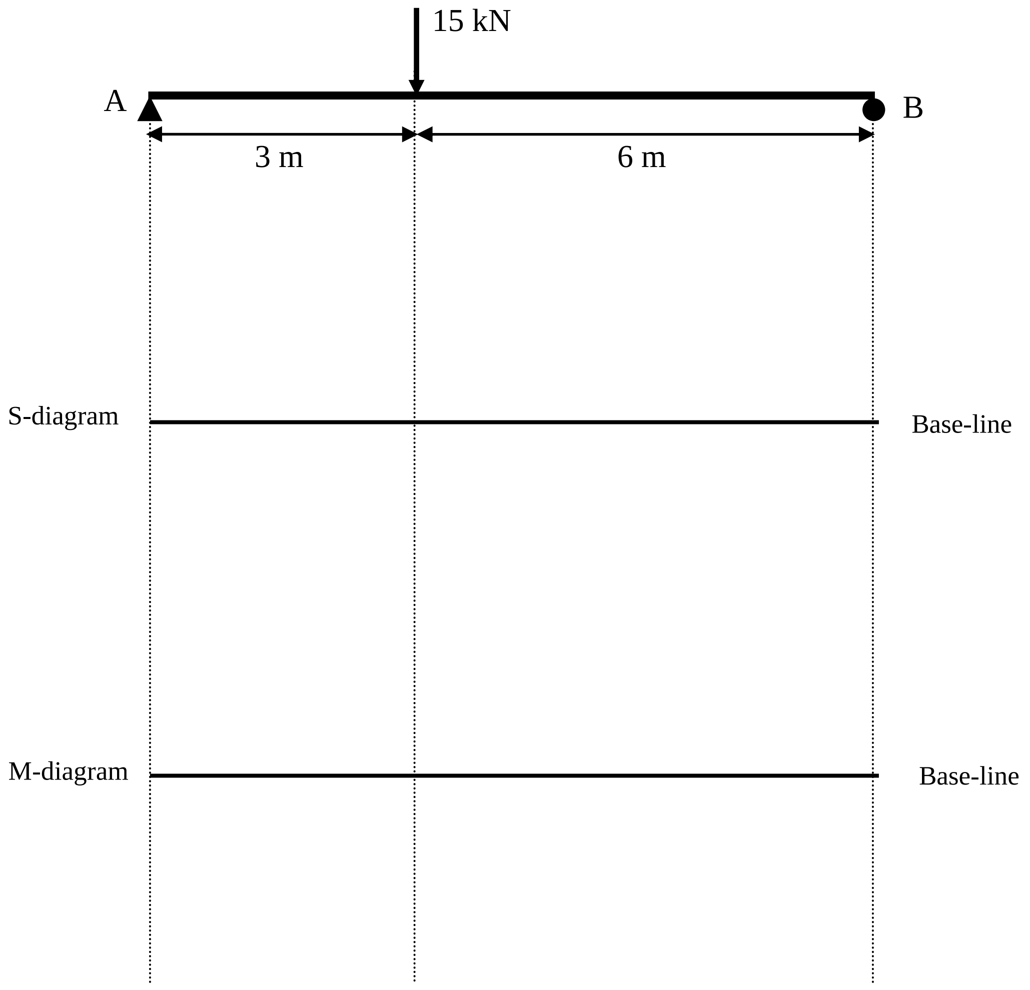
Q1. What is M at the point under the load?

Q2. What about S at that point?

Q3. What is M at the left or right support?

Q4. What about S at the supports?

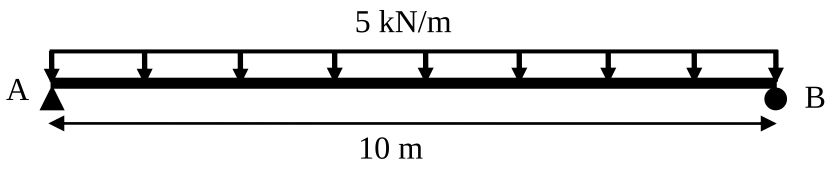
S and M Diagrams:



Note: Variation of S and M depends on support conditions and loading

What are “Strategic” points?

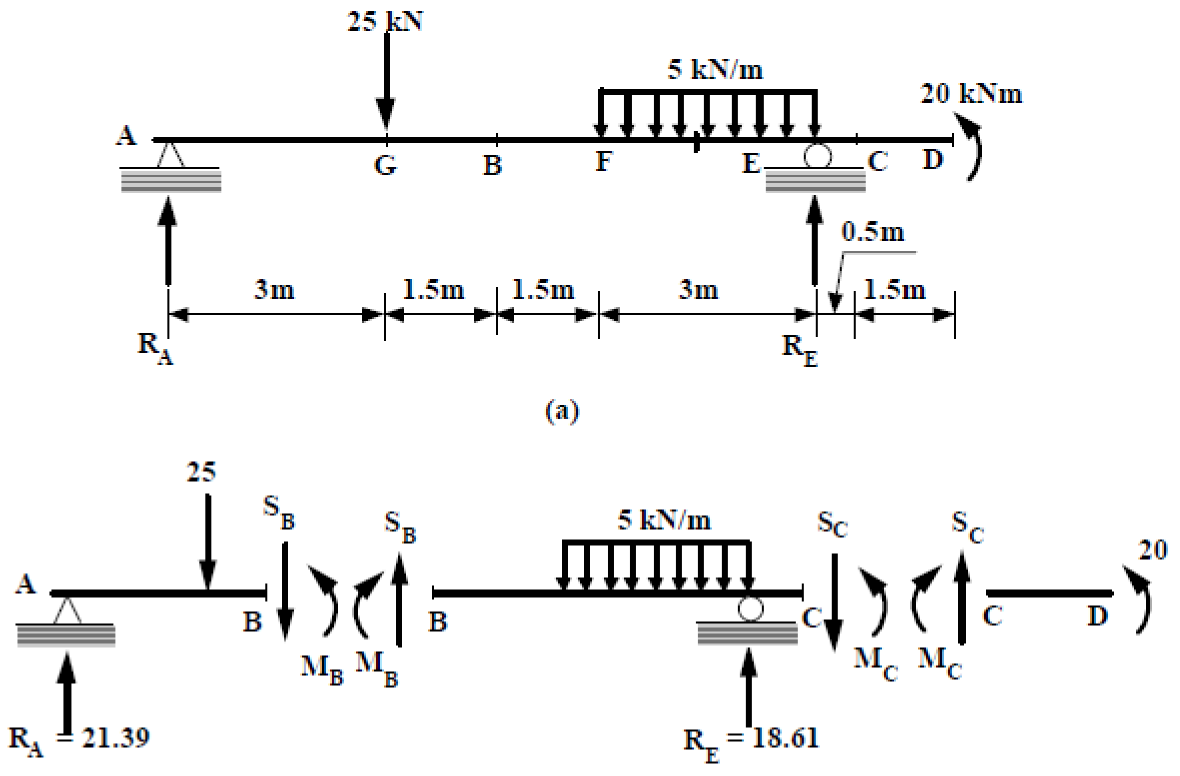
Example 2: Obtain the S and M values at the sections as shown then plot the S and M-diagrams for the beam under UDL.



Solution:

S and M Diagrams:

6.2 ILLUSTRATIVE EXAMPLE



(a)

(b) (c) (d)

6.3 RELATIONSHIPS BETWEEN LOADING, SHEAR AND MOMENT

6.3.1 Governing Equations

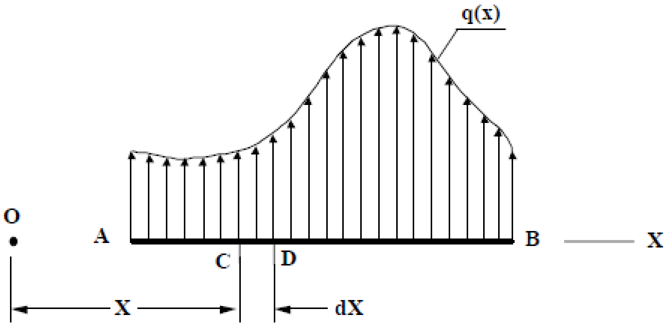
- The distributions of shear and bending moment along the span of the beam are affected by the loading conditions.

- S-Diagram (distribution of S from A to B – How?)
  M- Diagram (distribution of M from A to B – How?)

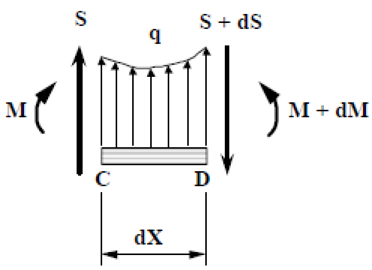
- What are S & M at strategic points (s/p) and at points in between s/p? so that curves may be plotted?

S

- Some mathematical analysis may help to gain an insight into the relationships between q, S and M, thus making the construction of S and M-diagrams easier.



(a)



(b)

$$\Sigma F_y = 0: \qquad S + q \cdot dx - (S + dS) = 0$$

or
$$q = \frac{dS}{dx}$$

Further,

$$\Sigma M \text{ at } D = 0: \qquad M - (M + dM) + S \cdot dx + q \cdot dx \frac{dx}{2} = 0$$

i.e.
$$-dM + S \cdot dx + q \frac{(dx)^2}{2} = 0$$

Note that the third term in the above equation is a second order quantity. Its magnitude is insignificant in comparison to the other two terms. Therefore, we can write

$$S = \frac{dM}{dx}$$

6.3.2 Interpretations

- It may be recalled from basic calculus that the first derivative of a mathematical function is, physically, the slope of the curve represented by that function.

- The loading at a given point along the beam is but the slope of the shear distribution curve at that point, with all its characteristics and features.

- The slope at a point of the bending moment distribution curve is the shear at that point. The slope gives the magnitude and direction of the shear at the given point.

6.4 SHEAR FORCE AND BENDING MOMENT DIAGRAMS

6.4.1 Significance:

- They give the full picture of the distribution of S and M values along the span of a beam.
- Knowing how to construct the S and M-diagrams means you have a full grasp of the behaviour of the beam. This is mandatory for a correct design of the beam.

Section 6.4.2 Illustrative Example

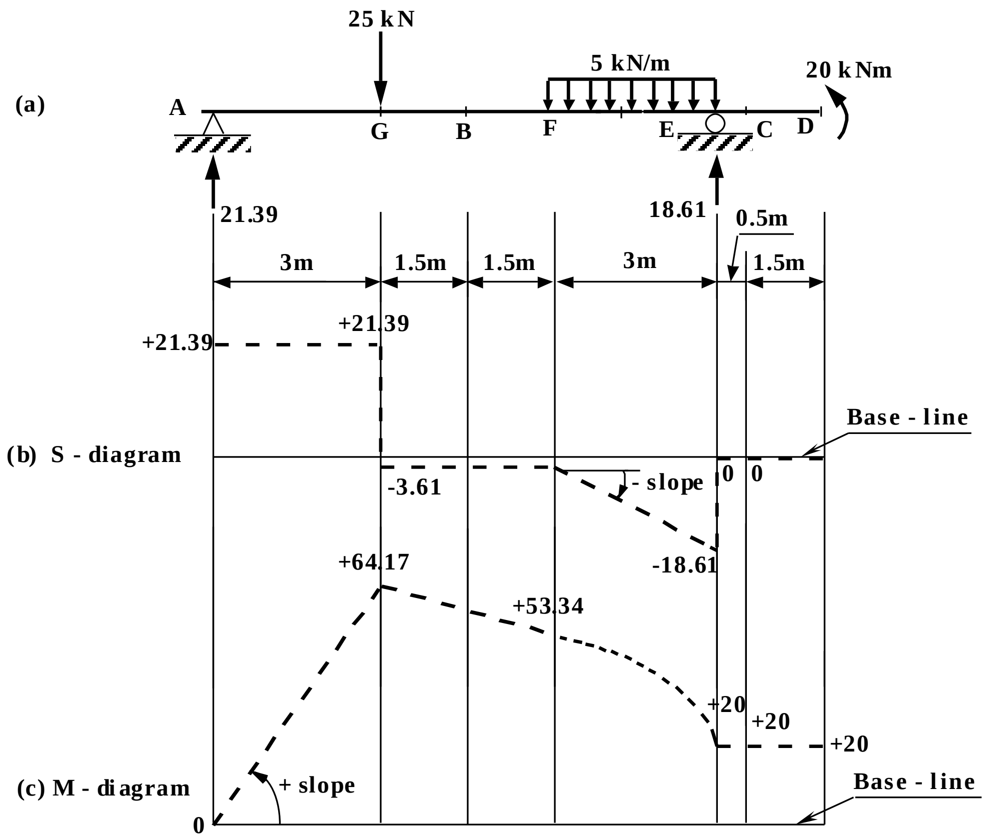
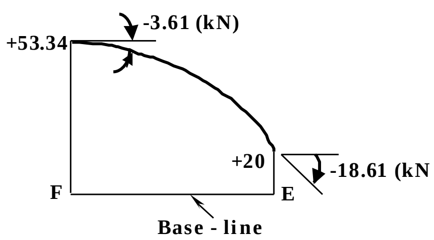
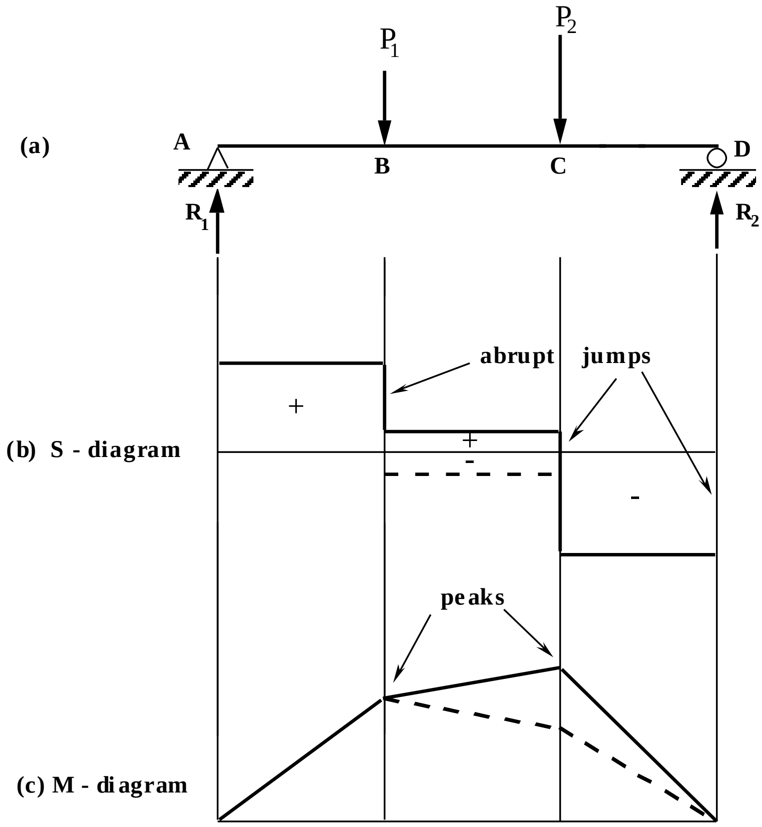


Fig. 8.7



6.4.3 Concentrated Forces



S-Diagram:

(i) Concentrated forces cause abrupt changes of shear values at points of applications (points A, B, C and D).

(ii) If there is no loading between two concentrated loads, the S-diagram at that length is a horizontal (straight) line.

(iii) The actual shear values at A, B, C and D depend on the magnitudes of the concentrated forces. Thus the segment between points B and C can either by (+) above or (-) below the base-line.

M-diagram:

(i) The M-diagram peaks at the points where applied concentrated loads act. The moment vanishes at exterior simple supports.

(ii) The moment varies linearly at segments of the beam where there are no external loads acting (in our case between A & B, B & C and C & D).

(iii) The actual moment values depend on the magnitudes and arrangements of the loads. At the segment between B and C, the M-diagram line may rise to give a maximum moment at C or it may decline to C with the moment at B as the maximum. Rise or fall depends on the shear diagram at this segment. If the shear diagram is positive then the M-diagram line rises; otherwise it falls as represented by the dotted lines.

(iv) Finally, moment is a maximum when shear diagram passes through zero from positive to negative.

6.4.4 Uniformly Distributed Loads

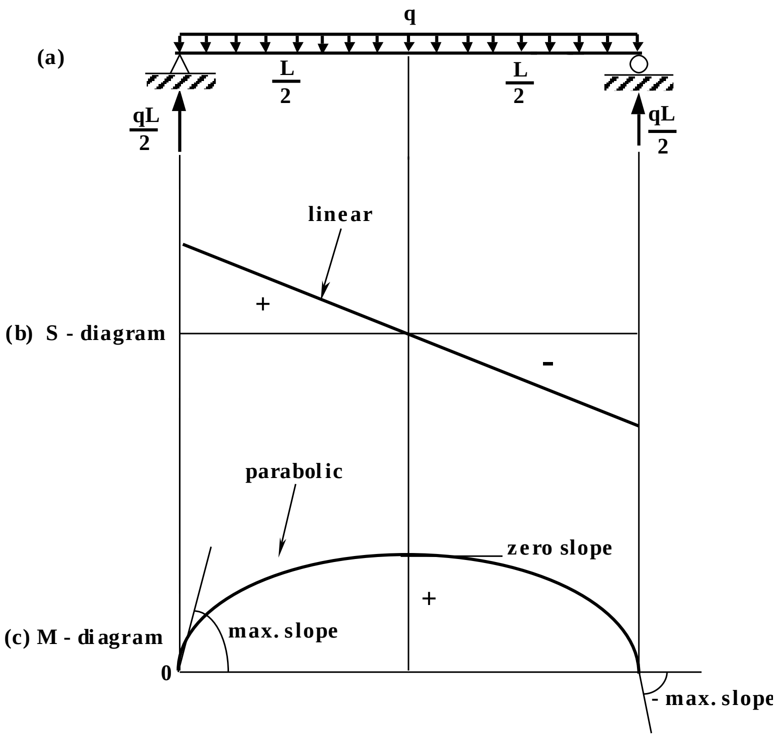
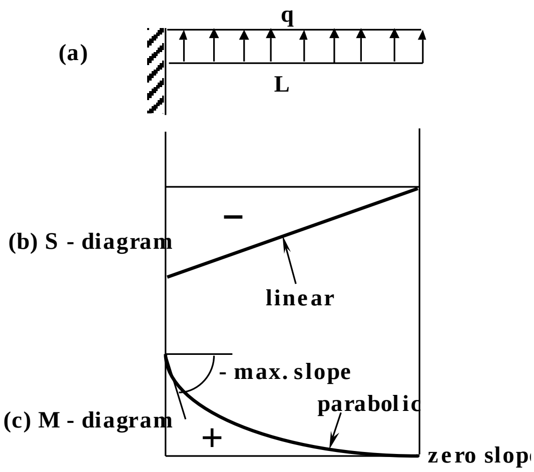


Fig. 8.10



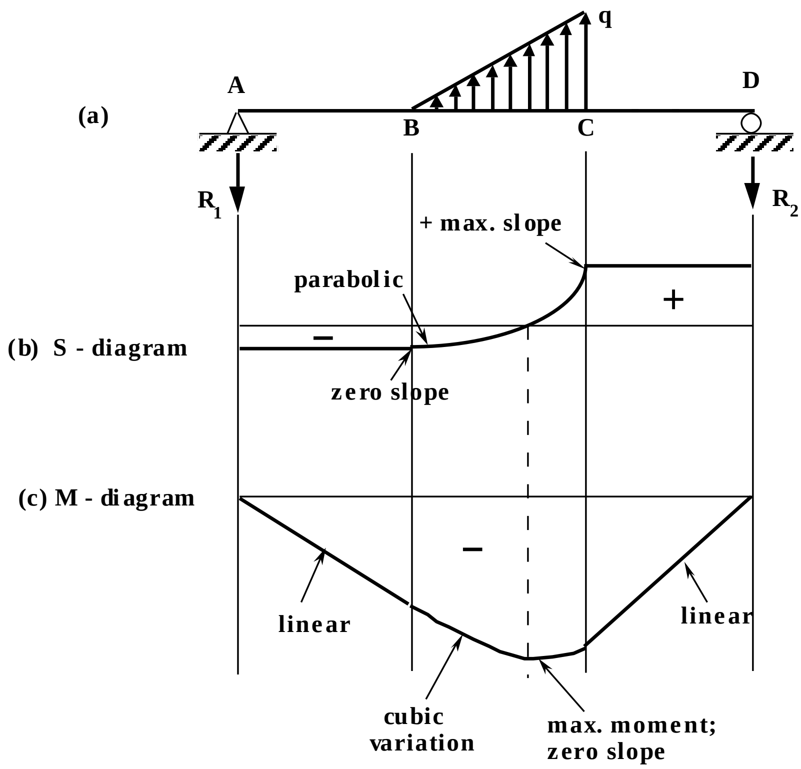
(i) Shear varies linear from end to end of the UDL.

(ii) Depending on the directions of the UDL, the constant slope of the S-diagram may be positive or negative: positive or upward load gives positive slope; downward load yields negative slope.

(iii) M-diagrams take the form of a parabola, the exact shape of which is influenced by the corresponding S-diagram. The maximum slope of the M-diagram occurs where the shear is a maximum; zero slope occurs where shear is zero.

(iv) Moment is a maximum where shear passes through zero from positive to negative.

6.4.5 Linearly Distributed Loads

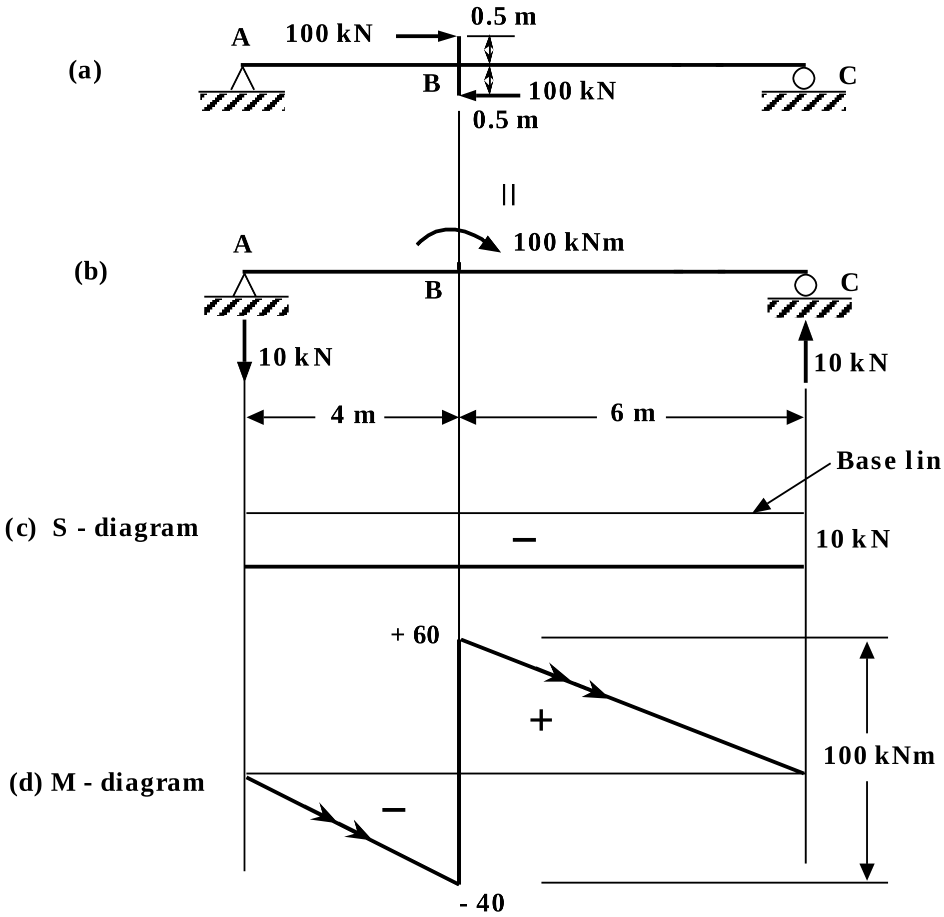


(i) Shear varies parabolically, and the moment variation is a cubic one, along where the LDL acts.

(ii) The shapes of the parabolic and cubic curves are dictated by the loading direction and the shear variation respectively.

(iii) Again, maximum moment occurs where the shear changes signs i.e. where the moment curve has a zero slope.

6.4.6 Concentrated Moments



(i) As there is no applied force between the supports, shear is constant throughout.

(ii) There is an abrupt change in the moment curve where the applied concentrated moment exists.

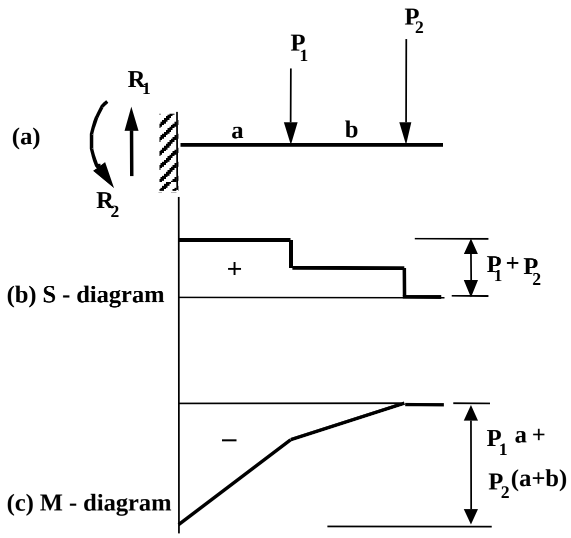
(iii) The moment variation is linear between points A and B and between B and C. The two lines are parallel to each other, i.e. they have the same slope. This is because the shear is constant over the whole span.

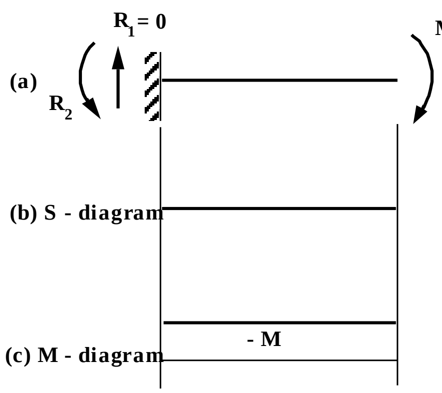
6.4.7 Simple Supports

- Simple supports are where concentrated reactions act. Thus their effects are similar to applied concentrated forces.
- At a simple support, either an exterior or interior one, there is an abrupt jump in the value of shear.
- At an exterior single support, the bending moment vanishes.
- For an interior simple support however the bending moment may assume any value as the reaction acts here just like any concentrated force.

6.4.8 Built-in Supports and Free Ends

- At a built-in or fixed support, there are the concentrated reaction force and the concentrated reaction moment. Thus shear and bending moment curves change abruptly from zero to the values equal to the reaction force and moment respectively.
- At the free end of a cantilever, both shear and moment vanishes except when it is subjected to a concentrated moment in which case shear remains zero but the bending moment is equal to the applied value.





<u>Major Steps for Constructing S and M-Diagrams</u>

(i) Compute the support reaction forces and moments (if any).

(ii) Compute S and M values at "**strategic**" points which include:

- support points (interior and exterior)
- points where concentrated forces and/or moments act
- limiting points of UDL, LDL etc
- free ends of cantilevers

(iii) Connect known values of S (for the S-diagram) and M (for the M-diagram) at adjacent strategic points using appropriate lines:

- a straight horizontal line;
- a straight line inclining upward or declining downward;
- a concave or convex parabolic/cubic curve;
- a higher order curve (which is highly unlikely unless the applied distributed load is parabolic or of a high order variation).

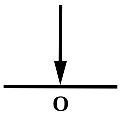
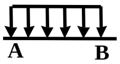
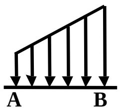
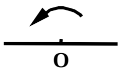
(iv) Check your work, as necessary, using the relationships:

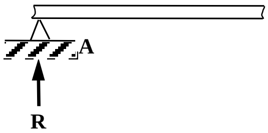
$$\frac{dM}{dx} = S$$

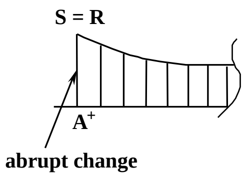
and

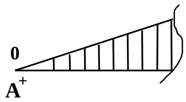
$$\frac{dS}{dx} = q$$

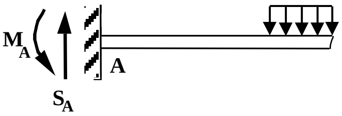
Features of S and M diagrams associated with loading and support types

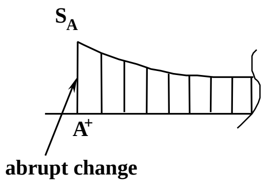
Loading / support		**Shear-diagram**	**Moment - diagram**
zero load		horizontal line	linear variation
Concentrated load		abrupt change at point of application	peaks at point of application
UDL		linear variation between A and B	parabolic variation between A and B; maximum at zero shear
LDL		parabolic variation between A and B	cubic variation between A and B; maximum at zero shear
Concentrated moment		no direct effects	Abrupt change at point of application

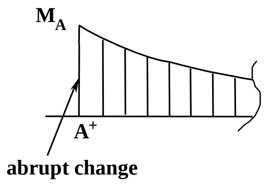


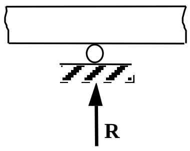






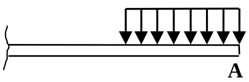


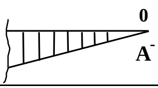


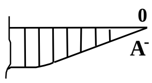


Same features as concentrated load (= R)

Same features as concentrated load (= R)



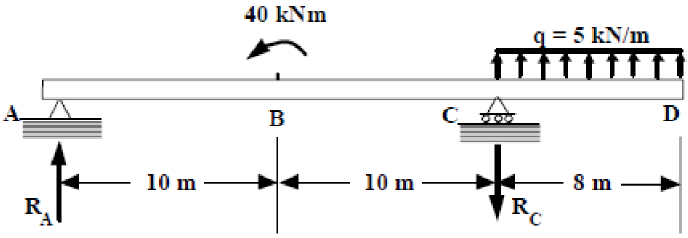




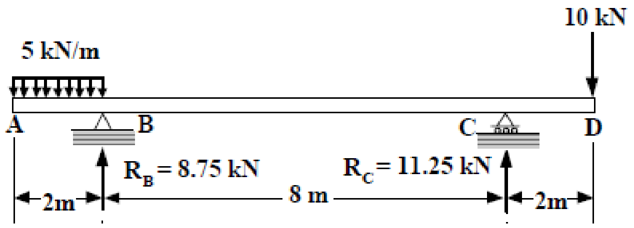
6.5 NUMERICAL EXAMPLES

6.5.1 Simply-Supported Beams with Overhangs

Example 1

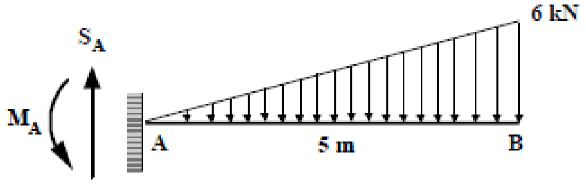


Example 2

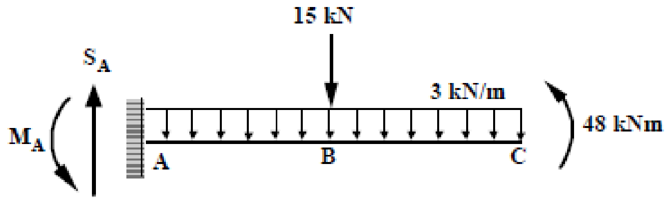


6.5.2 Cantilever Beams

Example 3



Example 4



CHAPTER 7 CENTROIDS AND CENTRES OF GRAVITY

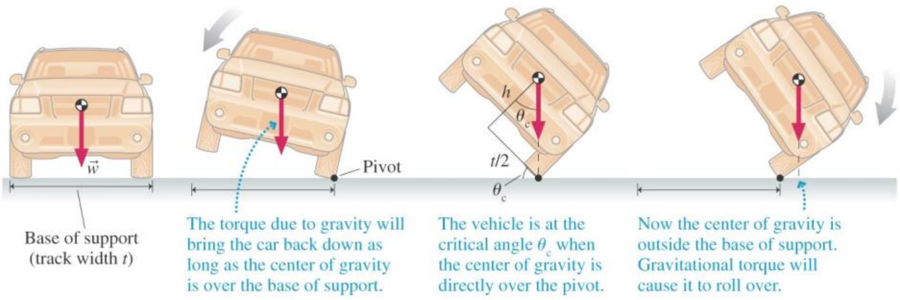
OBJECTIVES AND EXPECTED OUTCOMES

- *Understand the concepts of 'centroids' and 'centres of gravity'*
- *Determine centroids and centres of gravity of simple and composite areas*

7.1 DEFINITIONS AND TERMINOLOGY

7.1.1 General

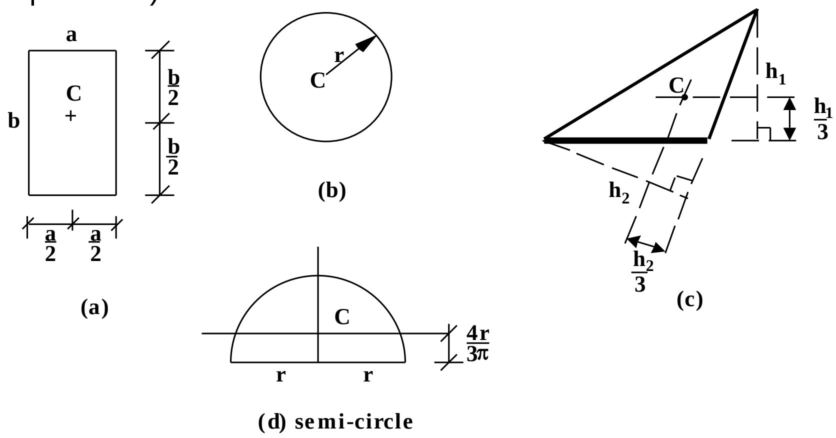
Stability of a car



7.1.2 Centroids

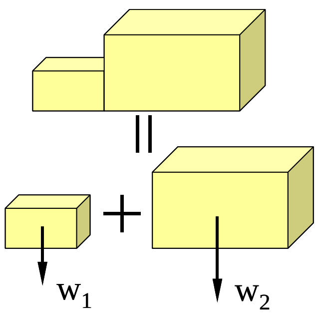
- Geometric centre of a plane/an area
- Pre-requisite for design calculations

<u>Examples</u> (simple areas)



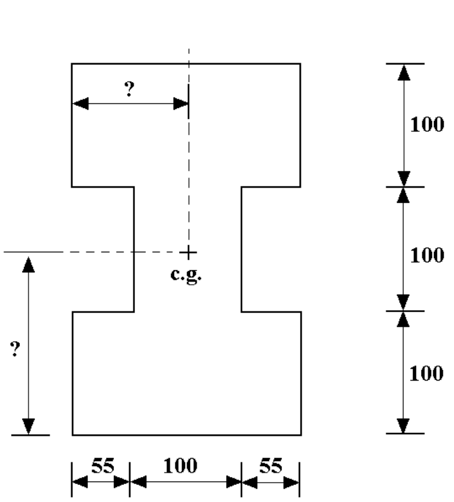
7.1.3 Centre of Gravity (c.g.)

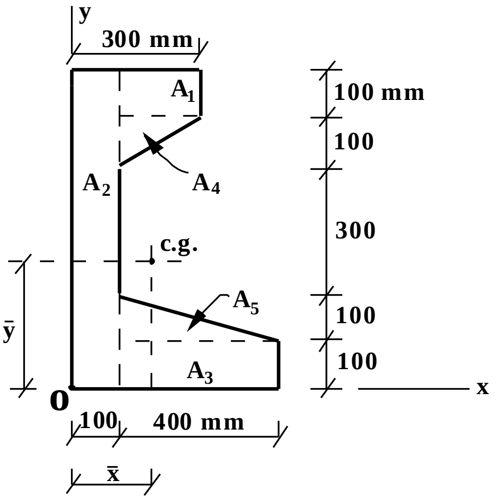
- It is where the resultant of a distributed force (over a line, area, and volume) acts.
- For a plane object of uniform thickness made up of a homogenous material, the centre of gravity location of the object coincides with the centroid of the plane.



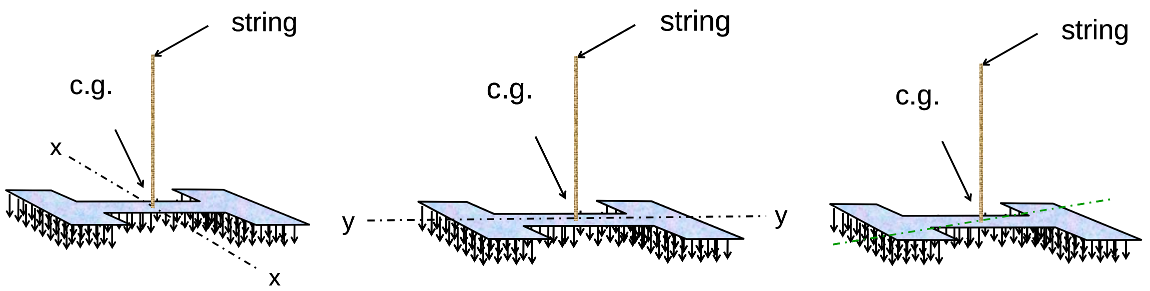
- Forces are involved;
- Identical to centroid for a planar object (area);
- For a solid/3D object or a group of parallel forces, it is where the resultant acts.

DEMOSTRATIONS





- There exists a unique point within or without the area of the plane from where the plane object (e.g. cardboard) would hang horizontally.
- This unique point is the **centre of gravity** of the said cardboard.
- **Summation of moments** due to all these distributed weights about any horizontal axis which passes through the centre of gravity is **zero.**

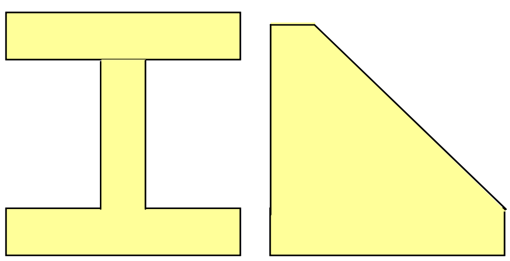


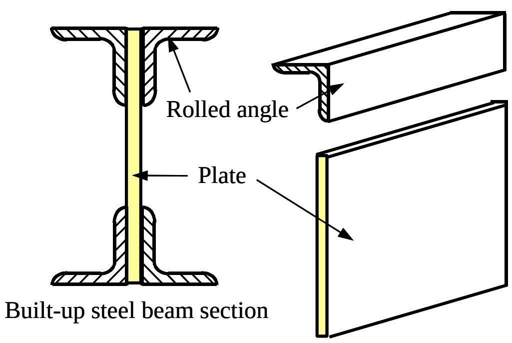
What do we need to master these topics?

- Simple statics concept/process – moment of area
- Equilibrium

7.2 COMPOSITE AREAS (making up of simple areas)

7.2.1 General





Composite area

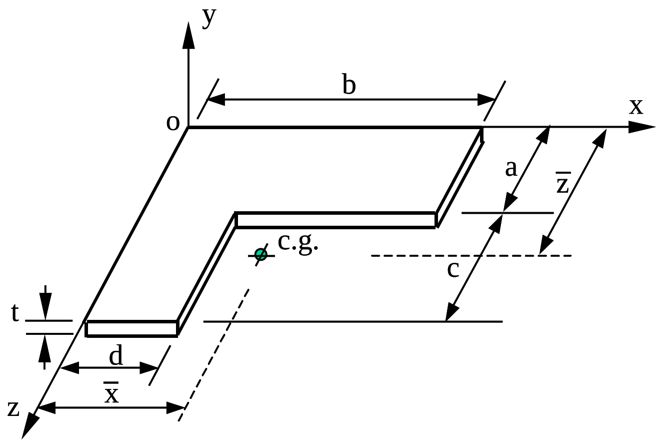
Compound area

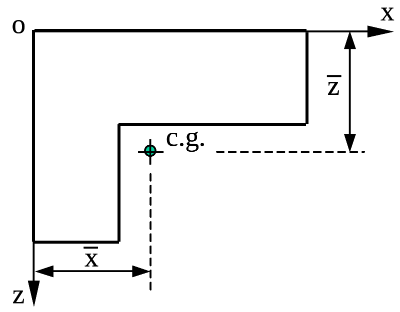
How to compute the centroid of a composite area?

7.2.2 Method of summation

Suitable for locating the c.g. of composite areas.

Example Given an L-shaped object with uniform thickness "t (m)" and weight density "γ (kN/m^3)", locate the c.g.





General Case

- Arbitrary-shaped plane object of a composite area
- Weight = **W**
- Object is divided into **n** simple parts
    - area $\mathbf{A_i}$ (m^2)
    - uniform thickness, $\mathbf{t_i}$ (m)
    - unit weight $\boldsymbol{\gamma_i}$ (kN/m^3)
    - centroid $\mathbf{c_i}$ (x_i, z_i)

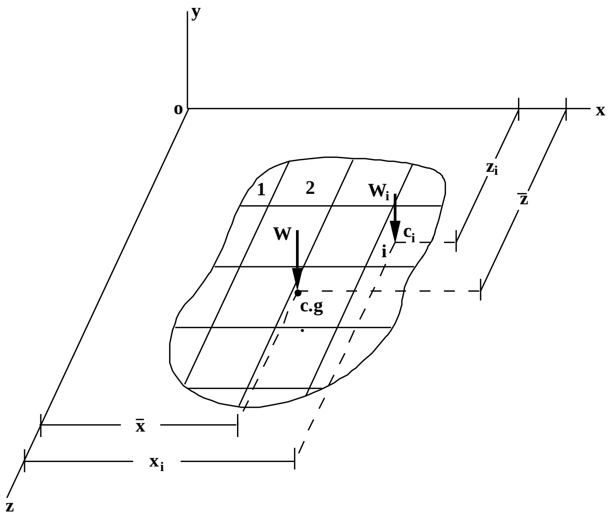


Fig. 6.3

Weight of A_i: $W_i = \Sigma\gamma_i\, t_i\, A_i$ . . . (6.1)

Moment about x axis: $M_{xi} = W_i\, z_i = \gamma_i\, t_i\, A_i\, z_i$ . . . (6.2)

Moment about z axis: $M_{zi} = W_i\, x_i = \gamma_i\, t_i\, A_i\, x_i$ . . . (6.3)

Total weight of object: $$W = \sum_{i=1}^{n} \mathbf{W_i} = \sum_{i=1}^{n} \gamma_i \mathbf{t_i A_i}$$

Total moment about x axis: $$\sum_{i=1}^{n} \mathbf{M_{xi}} = \sum_{i=1}^{n} \gamma_i \mathbf{t_i A_i z_i} \quad . . . (6.4)$$

Total moment about z axis: $$\sum_{i=1}^{n} \mathbf{M_{zi}} = \sum_{i=1}^{n} \gamma_i \mathbf{t_i A_i x_i} \quad . . . (6.5)$$

Centre of gravity (**c.g.**) — the point at which the resultant acts

Moment about **z** axis due to **W**: $$\mathbf{W}\,\bar{x} = \sum_{i=1}^{n} \mathbf{M_{zi}} \quad . . . (6.6)$$

Location of **c.g.**: $$\bar{\mathbf{X}} = \frac{\sum_{i=1}^{n} (\gamma_i \mathbf{t_i A_i x_i})}{\sum_{i=1}^{n} (\gamma_i \mathbf{t_i A_i})} \quad . . . (6.7)$$

$$\bar{Z} = \frac{\sum_{i=1}^{n} (\gamma_i \mathbf{t_i A_i z_i})}{\sum_{i=1}^{n} (\gamma_i \mathbf{t_i A_i})} \quad . . . (6.8)$$

For uniform thickness **t** and homogeneous material with weight density γ_i,

$$t_1 = t_2 = \ldots\ldots = t_i = \ldots\ldots = t_n = t \qquad \ldots (6.9)$$

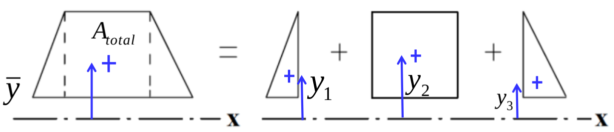
and $$\gamma_1 = \gamma_2 = \ldots\ldots = \gamma_i = \ldots\ldots = \gamma_n = \gamma \qquad \ldots (6.10)$$

Location of **c.g.**: $$\bar{X} = \frac{\sum_{i=1}^{n}(A_i x_i)}{\sum_{i=1}^{n}(A_i)} = \frac{\sum_{i=1}^{n}(A_i x_i)}{A} \qquad \ldots (6.11)$$

and $$\bar{Z} = \frac{\sum_{i=1}^{n}(A_i z_i)}{A} \qquad \ldots (6.12)$$

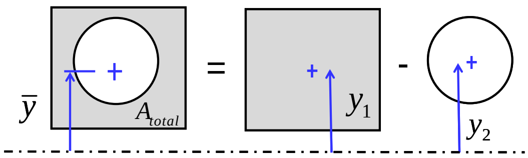
Procedure involves summations of quantities — **method of summations**

$$\bar{y} \times A_{total} = y_1 \times A_1 + y_2 \times A_2 + y_3 \times A_3$$

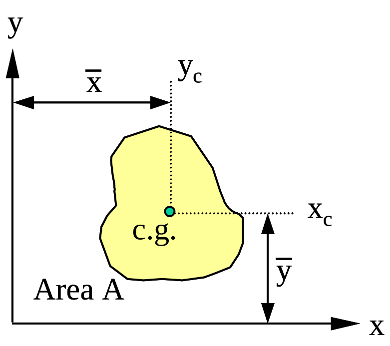


$$\bar{y} = \frac{\sum_{i=1}^{n}(A_i y_i)}{A}$$

$$\bar{y} \times A_{total} = y_1 \times A_1 - y_2 \times A_2$$



7.2.3 First moment of an area $\bar{\mathbf{Q}}$



$$\bar{\mathbf{Q}}_x = \mathbf{A} \cdot \bar{\mathbf{y}} \quad \text{and} \quad \bar{\mathbf{Q}}_y = \mathbf{A} \cdot \bar{\mathbf{x}}$$

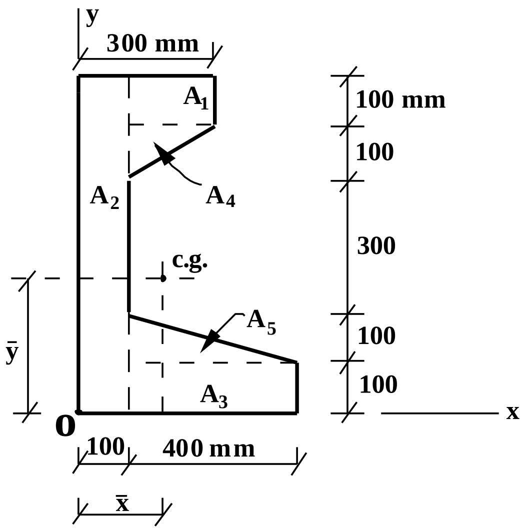
$$\bar{\mathbf{Q}}_{xc} = \mathbf{A} \cdot (0) = 0 \text{ and}$$

$$\bar{\mathbf{Q}}_{yc} = \mathbf{A} \cdot (0) = 0$$

First moment of area w.r.t. its centroidal axes (x_c and y_c)= 0

7.2.4 Numerical examples

Example 7.1 Figure below shows the detailed dimensions of the cross section of a concrete beam. Locate the centroid of the section with respect to the corner point O.



The section is a composite area consists of three rectangles A_1, A_2 and A_3, and two triangles A_4 and A_5.

Total area of the section:

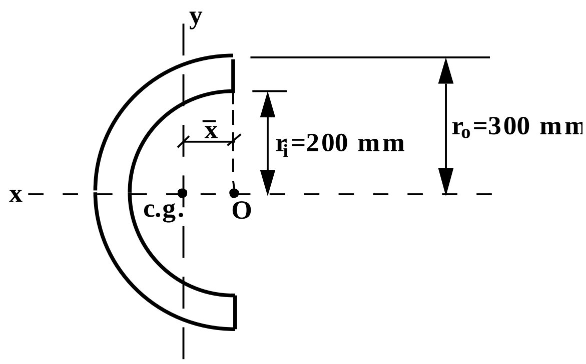
$A = \sum_{i=1}^{5} A_i = 200\times100 + 100\times700 + 400\times100 + 100\times200/2 + 100\times400/2$

$= 160 \times 10^3$ mm^2

Solution:

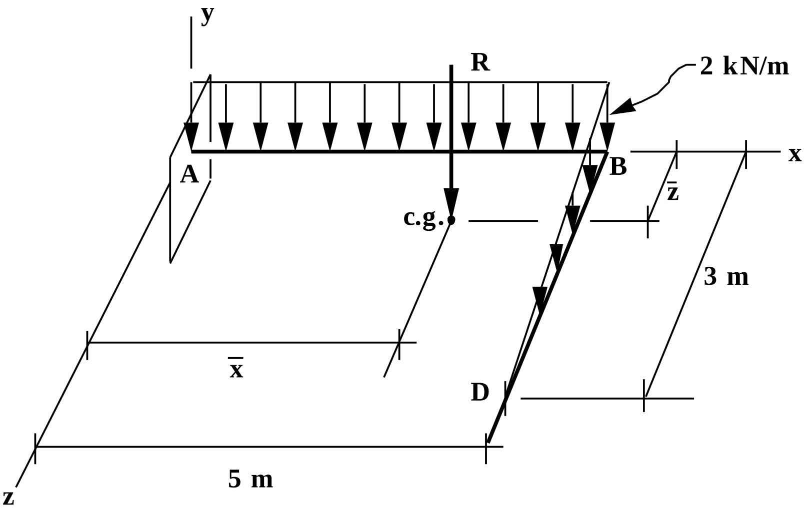
Shape	A (mm^2)	x (mm)	y (mm)	Ax (mm^3)	Ay (mm^3)

Example 7.2 For the semi-circular thin-walled beam section shown in figure below, determine the position of the centroid with respect to the origin of the circles, O.



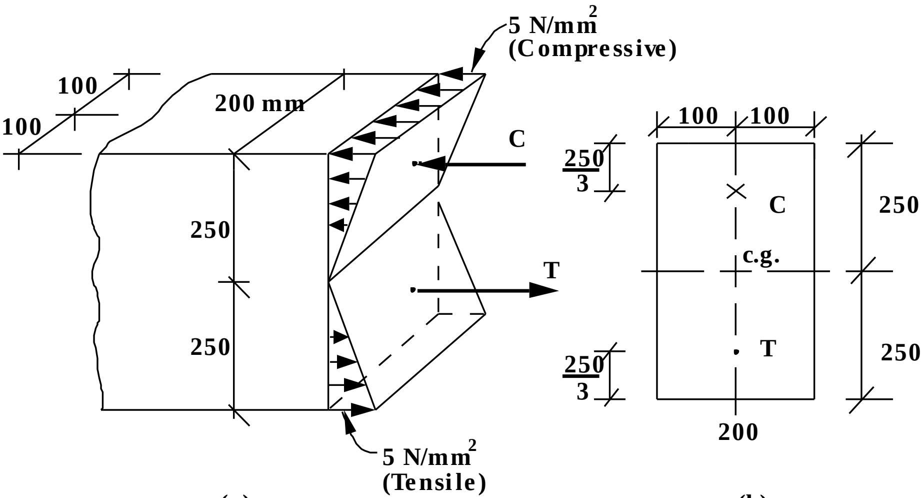
Solution:

Example 7.3. A cantilever bent is loaded laterally as shown in figure below by a distributed load. Determine the location of the resultant R with respects to the support A.



Solution:

Example 7.4 Figure below illustrates the distribution of internal stress (force per unit area) over the cross section of a rectangular beam. Compute the resultants C and T and determine their locations.



(a) (b)

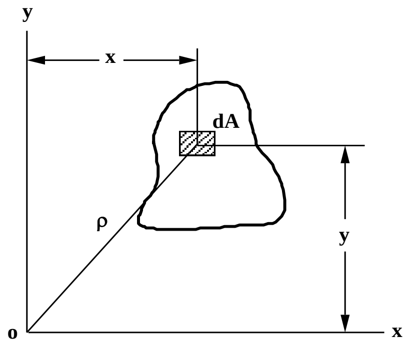
Solution:

CHAPTER 8 MOMENTS OF INERTIA
(Second Moments of Area)

OBJECTIVES AND EXPECTED OUTCOMES

- *Understand the concept of second moment of area*
- *Determine second moments of area of simple and composite areas*
- *Understand principle of parallel axes*
- *Understand engineering application in later courses*

8.1 DEFINITIONS



$$I_x = \int_A y^2 dA \qquad \text{(w.r.t. x-axis)} \qquad \ldots (7.1)$$

$$I_y = \int_A x^2 dA \qquad \text{(w.r.t. y-axis)} \qquad \ldots (7.2)$$

$$J = \int_A \rho^2 dA \qquad \text{(w.r.t. z-axis, or polar moment of inertia)} \qquad \ldots (7.3)$$

$$J = \int_A (x^2 + y^2) dA = I_x + I_y \qquad \ldots (7.4)$$

Origin

The expressions are resulted from the derivation of bending and torsion stress analysis formulas for beams. It is always required w.r.t. a given axis (i.e. x, y or z)

First moment of area

$$Q_x = \int_A y dA = A \cdot \bar{y}$$

$$Q_y = \int_A x dA = A \cdot \bar{x}$$

I ≡ second moment of area. A mathematical quantity measuring the efficiency of a shape in respect of its resistance to bending.

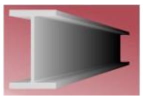
Symbols and units

I, I_x, I_y, J

Applications

Required in all calculations leading to stresses in, and deformations of, structures and structural members.

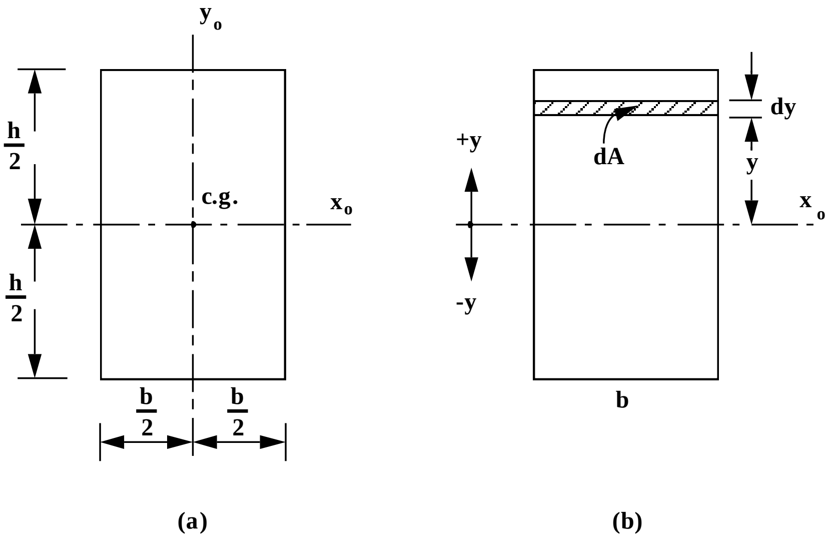
Stress analysis $\sigma = \frac{My}{I}$



Fluid mechanics $y_{cp} = \bar{y} + \frac{\bar{I}}{\bar{y}A}$

8.2 SIMPLE CROSS SECTIONS

8.2.1 Rectangular Section



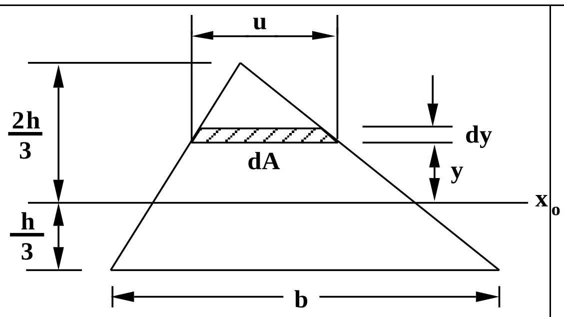
$$I_{xo} = \int_A y^2 dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b\,dy = \left[\frac{by^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{bh^3}{12} \qquad \text{. . . (7.5)}$$

Similarly,

$$I_{yo} = \frac{hb^3}{12} \qquad \text{. . . (7.6)}$$

- Unit for I: **mm^4** (not to use cm^4, m^4)
- I is always +Ve (*first moment of area can be ±Ve*)
- I is meaningful only when it is w.r.t a given axis, because I varies from axis to axis

8.2.2 Triangular Section



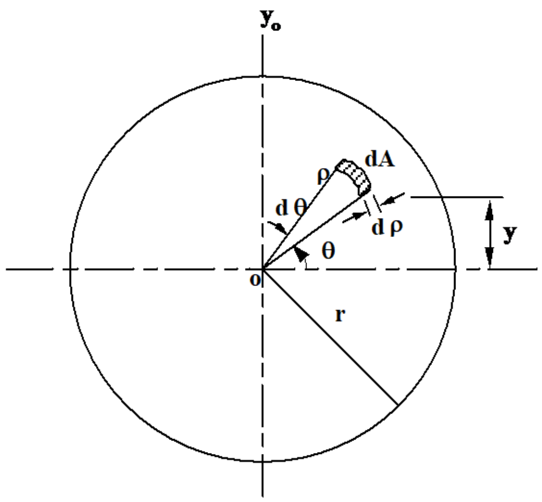
$dA = u\,dy$

$$\frac{u}{b} = \frac{\frac{2h}{3} - y}{h} \quad \therefore u = \frac{2b}{3} - \frac{b}{h} y$$

Using Eq. 7.1

$$I_{xo} = \int_{-\frac{h}{3}}^{\frac{2h}{3}} y^2 \left(\frac{2b}{3} - \frac{b}{h} y\right) dy = \frac{bh^3}{36} \quad \text{.. (7.7)}$$

8.2.3 Circular Section



I_X can be conveniently obtained using Eq. 7.1 by adopting the polar coordinate system.

The shaded area $dA = (\rho\, d\theta) \cdot d\rho$
and $y = \rho \sin\theta$

Using Eq.7.1:

$$I_{xo} = \int_0^r \int_0^{2\pi} \rho^2 \sin^2\theta \rho d\theta d\rho = \frac{\pi r^4}{4} \quad \text{. . . (7.8)}$$

Needless to say,

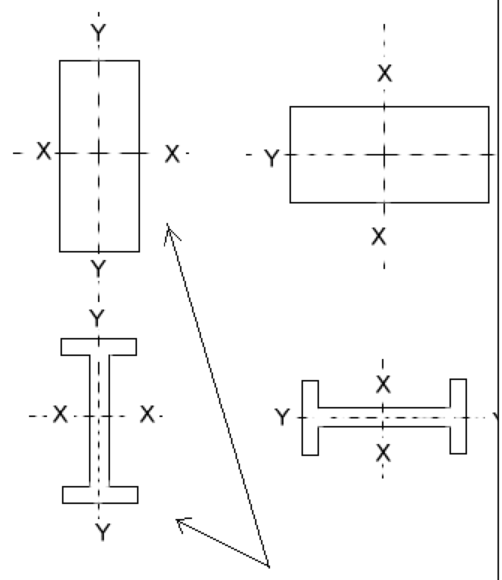
$$I_{yo} = I_{xo} = \frac{\pi r^4}{4} \quad \text{. . . (7.9)}$$

Polar moment of inertia according to Eq. 7.3:

$$J_o = \int_A \rho^2 dA = \int_0^r \int_0^{2\pi} \rho^2 \rho d\theta d\rho = \frac{\pi r^4}{2}$$

Confirming $J_o = I_{xo} + I_{yo}$

Understanding Second Moment of Area

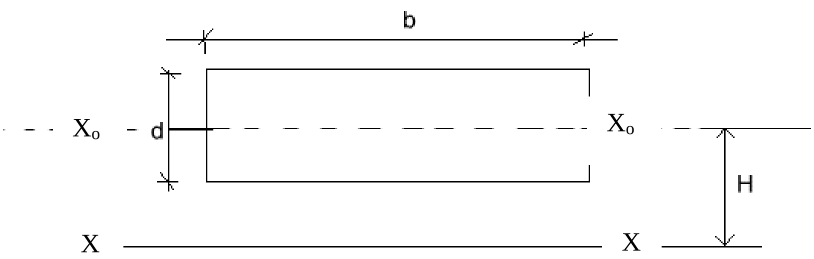
<table>
<tr><td>• Second moment of area (or moment of inertia) is a property which measures the efficiency of that shape in respect of its resistance to bending.</td><td rowspan="4">

This orientation is more efficient in resisting bendin</td></tr>
<tr><td>• Certain shapes are better able to resist bending than others and, in general, a shape is more efficient when the greater part of its mass is as far as possible from its centroid (or c.g.).</td></tr>
<tr><td>• Second moments of area about axes are usually termed I_X and I_Y.</td></tr>
<tr><td>• For non-symmetrical sections these values are different. The larger the second moment of area the better the resistance to bending.</td></tr>
</table>

Principle of Parallel Axes

- When a section is composed of more than one rectangle, "I" for the whole section can be obtained by summating the "I" values for the individual sections providing the sections all have a common principal axis.
- However, where individual sections do not have a common principal axis but their individual axes are parallel, the principle of parallel axes must be applied.

$$I_X = I_{XO} + AH^2$$

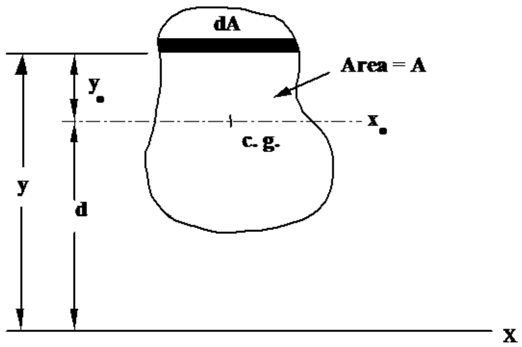


Where X-X = principal axis for the whole section
X_o-X_o = principal axis for that particular section
A=section area = bd
H=distance between axes X-X and X_o-X_o

8.3 TRANSFER FORMULAS FOR PARALLEL AXES

Applications

In practice it is always necessary to determine I or J w.r.t. axes parallel to x_o and y_o (centroidal/principal axes).



$$I_x = \int_A y^2 dA \qquad \text{. . . (a)}$$

But $y = y_o + d$ . . . (b)

$$I_x = \int_A y_o^2 dA + 2d\int_A y_o dA + d^2 \int_A dA \qquad \text{. . . (c)}$$

Note $2d\int_A y_o dA = 2d \cdot Q_x = 2d \cdot (A \cdot \bar{y}) = 2d \cdot (A \cdot 0) = 0$

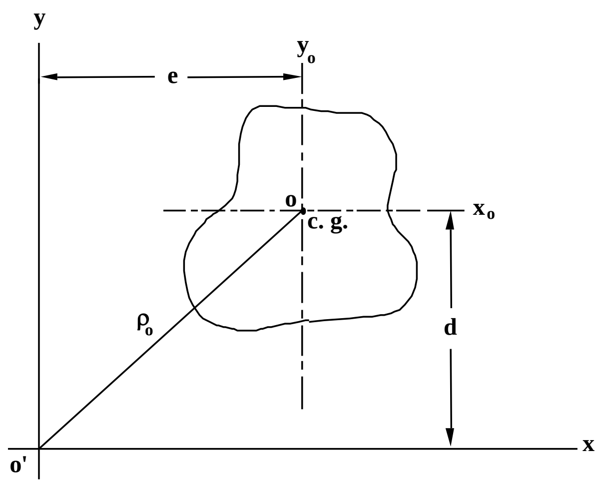
$$\boxed{I_x = I_{xo} + Ad^2} \qquad \text{. . . (7.11)}$$

Similarly

$$\boxed{I_y = I_{yo} + Ae^2} \qquad \text{. . . (7.12)}$$

- Parallel-axis theorem - highly important pair of equations
- The axes between which the transfer is made must be parallel
- One of the axes must pass through c.g. of the area
- $I_x \geq I_{xo}$ and $I_y \geq I_{yo}$

Polar moment of inertia w.r.t. o'

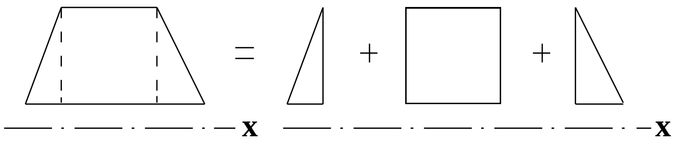


$J = I_x + I_y$
$\quad = I_{xo} + Ad^2 + I_{yo} + Ae^2$
$\quad = I_{xo} + I_{yo} + A(d^2 + e^2)$

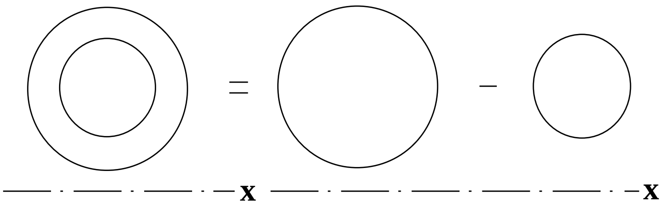
$J = J_o + A\,\rho_o^2$ . . . (7.13)

where J_o is referred to the c.g. of the section.

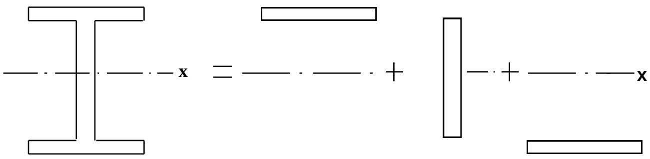
8.4 COMPOSITE AREAS



(a) Trapezoid

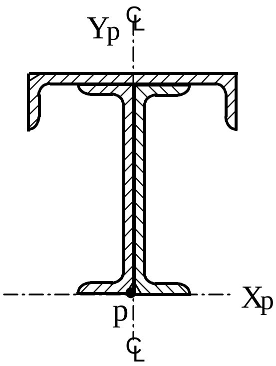


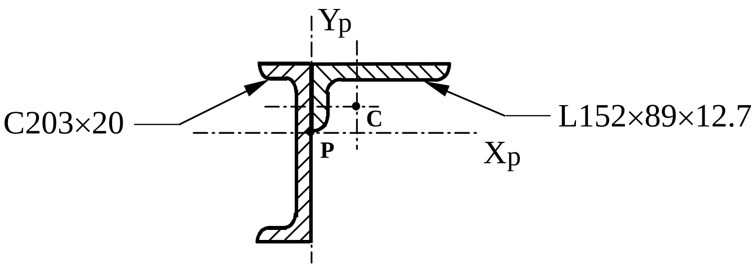
(b) Pipe



(c) I - section

Compound sections





Analysis steps:

Step 1: Locate x_o, y_o

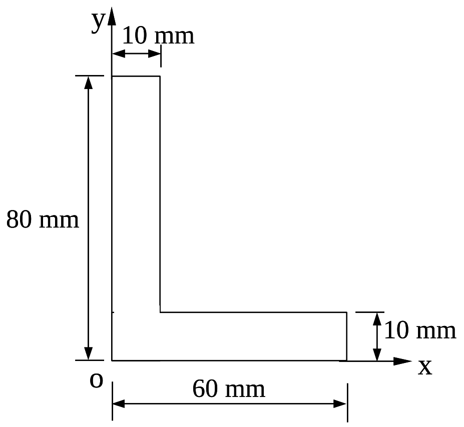
Step 2: Apply transfer equations to each section

Step 3: Sum up contributions from each section to obtain I_{xo} and I_{yo} for the composite/compound section

Example

For the unequal-leg angle,

(1) Determine the location of the centroid "C" with respect to point O.
(2) Compute the moments of inertia w.r.t. the centroidal axes X_C and Y_C.



8.5 RADIUS OF GYRATION

8.5.1 Definition

- In some structural engineering and mechanics computations it is required to determine the quantity called radius of gyration of the cross section of a structural member.
- It is required in column design and solutions to some dynamics problems.
- It describes the way in which the area of a cross-section is distributed around its centroidal axis. It is useful in estimating the stiffness of a column (buckling analysis).

  If the area is concentrated far from the centroidal axis it will have a greater value of $\bar{r}$ and a greater resistance to buckling.
- It is a mathematical quantity, not a distance that can be directly measured.

$$\bar{r}_x = \sqrt{\frac{I_x}{A}} \text{ (mm)} \qquad \bar{r}_y = \sqrt{\frac{I_y}{A}} \qquad \bar{r}_z = \sqrt{\frac{J}{A}}$$

$$\bar{r}_z^{\,2} = \bar{r}_x^{\,2} + \bar{r}_y^{\,2}$$

PART 2 – TUTORIAL QUESTIONS

Week 1 Tutorial Questions

CHAPTER 1: FUNDAMENTALS

Example 1:

Three forces are applied to a bracket as shown in Fig. 1. Determine

(a) The moment of force F_C about point B.
(b) The moment of force F_D about point A.
(c) The moment of force F_B about point C.

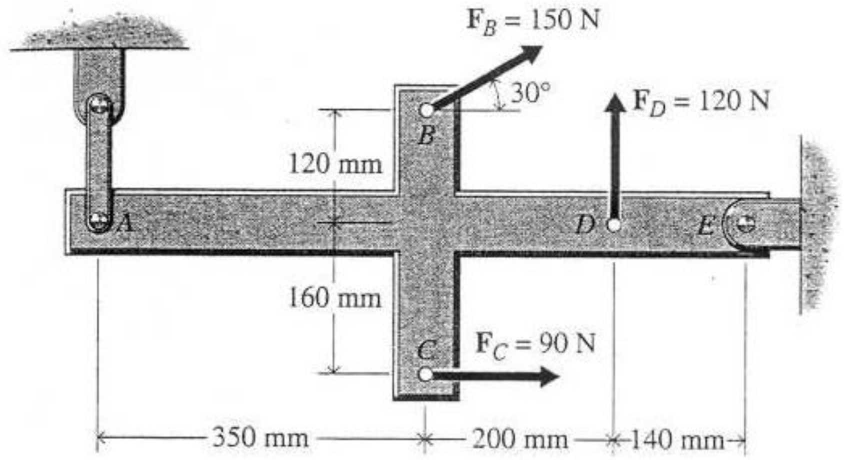


Fig. 1

Example 2:

A 1000N force is applied to a beam cross section as shown in Fig. 2. Determine the moment of the force about point O.

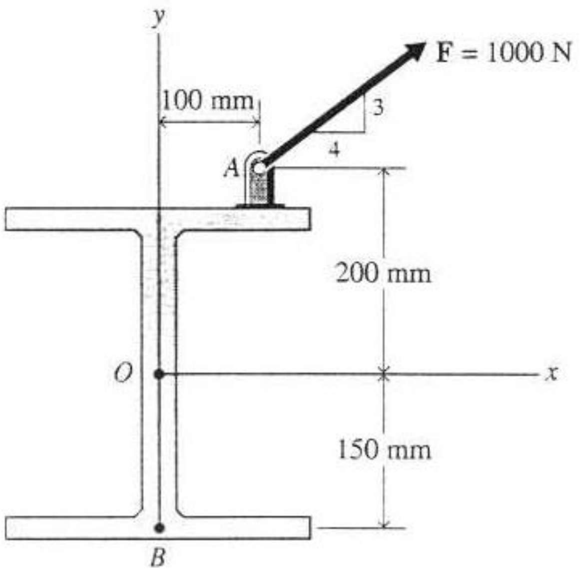


Fig. 2

Week 2 Tutorial Questions

CHAPTER 2: EQUILIBRIUM EQUATIONS

Example 1:

A planar system is shown in Fig. 1, set up, quantitatively, the relevant equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$. Note that the moment equations should be with respect to point O as indicated. Also compute F_1, F_2 and F_3.

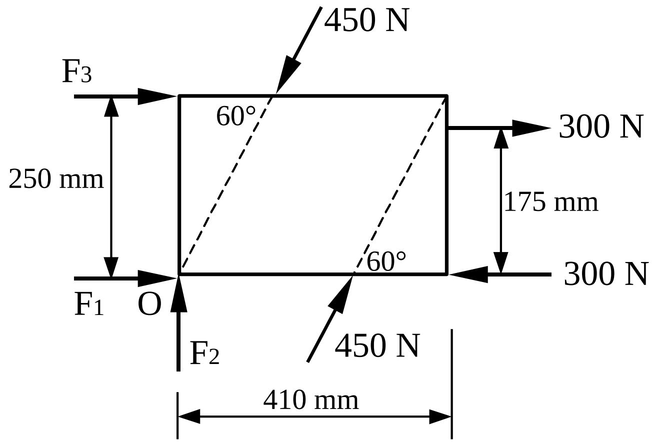


Fig. 1

Example 2:

Two forces and a moment produced by a couple of forces are applied to a bracket as shown in Fig. 2. Set up, quantitatively, the relevant equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$. Note that the moment equations should be with respect to point O as indicated. Also compute F_1, F_2 and F_3.

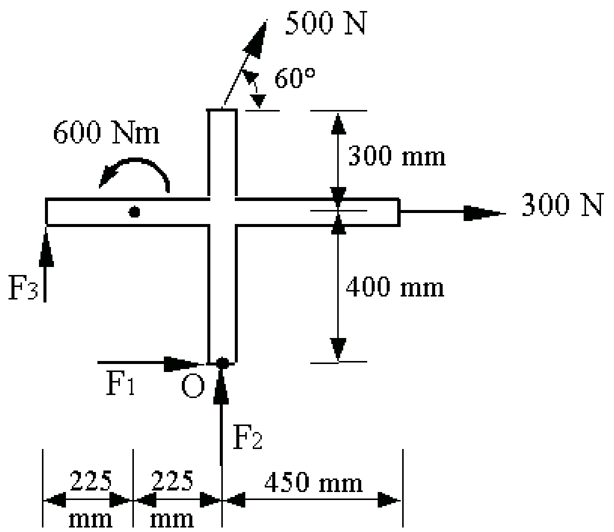


Fig. 2

Week 3 Tutorial Questions

CHAPTER 3: SUPPORTS AND SUPPORT REACTIONS

Example 1:

Inspect the "structures" illustrated in Figs. 1(a), (b), (c) and (d), make necessary calculations and answer, for each, the following questions:

(i) Stable?
(ii) If "no", why?
If "yes", statically determinate or indeterminate?

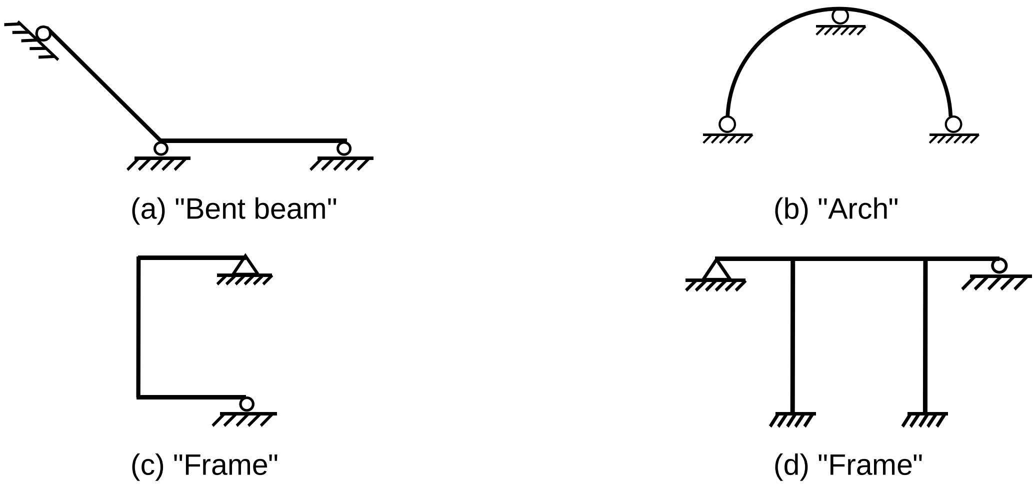


Fig. 1

Example 2:

Fig. 2 shows details of a simply-supported beam with overhangs. Compute the support reactions and indicate their directions.

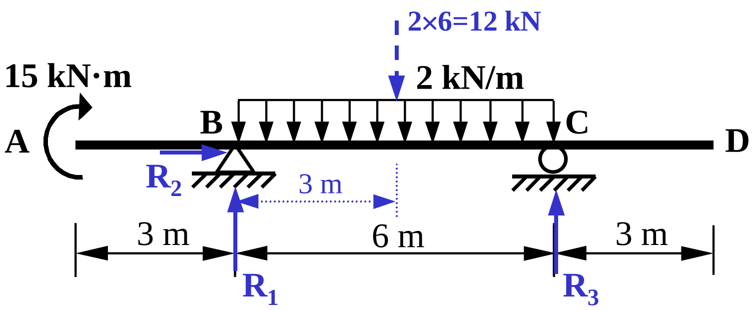


Fig. 2

Example 3:

Fig. 3 illustrates a three-hinged semi-circular arch under uniformly distributed load. Compute the reactions at supports **A** and **B**, and indicate their directions.

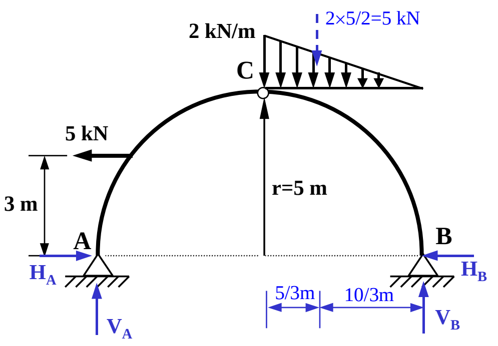


Fig. 3

Week 4 Tutorial Questions

CHAPTER 4: ANALYSIS OF TRUSSES

Example 1:

A truss is detailed in Fig. 1. For the given loading:

(a) Identify the null member(s) and the null joint(s).

(b) Determine the magnitude and nature of the axial force in members identified as **1**, **2** and **3**.

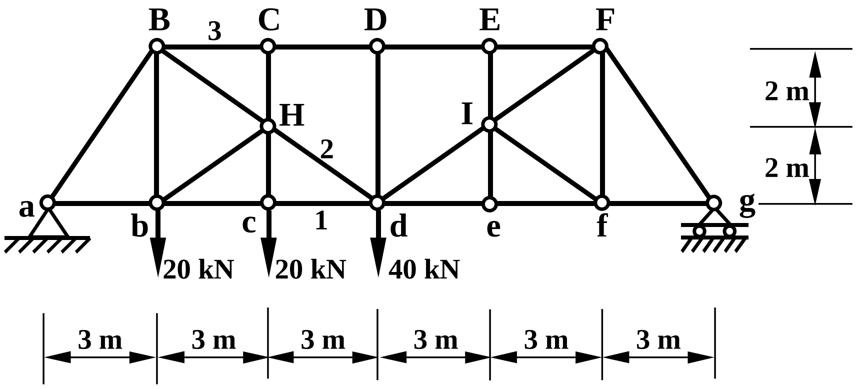


Fig. 1

Weeks 8 and 9 Tutorial Questions

CHAPTER 8: SHEAR FORCE AND BENDING MOMENT IN BEAMS

For each of the beams detailed below, construct the shear force and bending moment diagrams under the given loading. The values at the "strategic" points must be given.

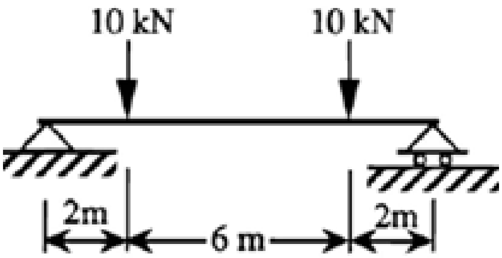


Fig. 1

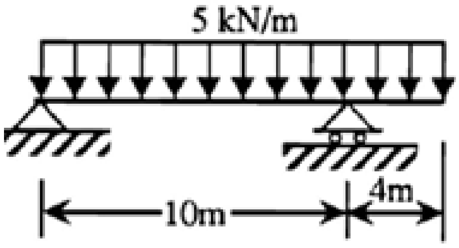


Fig. 2

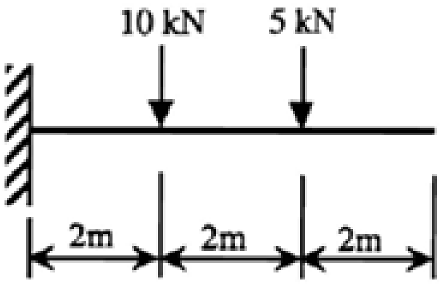


Fig. 3

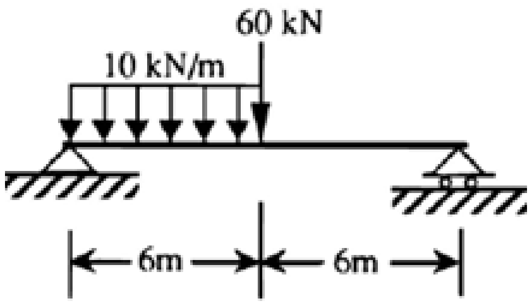


Fig. 4

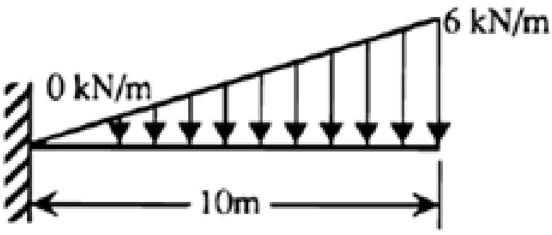


Fig. 5

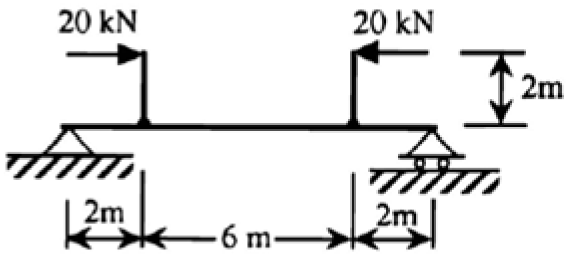


Fig. 6

Week 10 Tutorial Questions

CENTROIDS AND MOMENT OF INERTIA

Example 1:

A composite section is detailed in Fig. 1.

(a) Determine the location of the centroid with respect to point **O**;

(b) Determine the moments of inertia with respect to the X and Y axes;

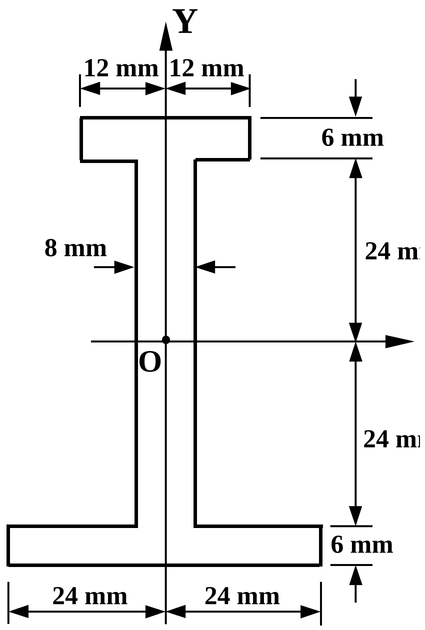


Fig. 1

PART 3 – FORMATIVE QUESTIONS

FUNDAMENTALS

Q.1

Determine the magnitude of the *u*- and *v*-components of the 900N force shown in Fig. 1.

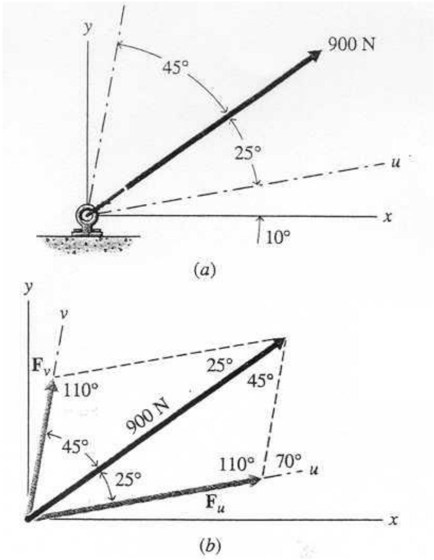


Fig. 1

Q.2

In Fig. 2 shown below, F_1 = 50 kN. Compute the unknown component forces F_2, F_3 and F_4 with the given directions.

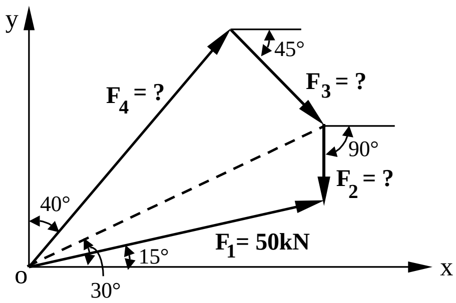


Fig. 2

Q.3

A force F and a monment M are applied to a beam as shown in Fig. 3. Determine the magnitude and direction of the moment with respect to point A.

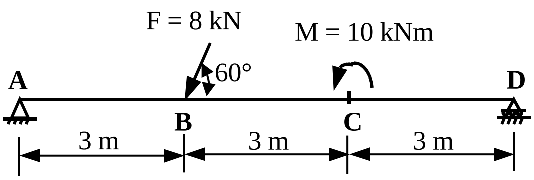


Fig. 3

Q.4

In each of the two figures in Fig. 4, a force is defined. Compute the unknown component forces with the given directions.

Hint: *Determine components.*

(a) $F_1 \rightarrow F_2$, F_3 (Note that the arrows representing F_1 and F_2 indicate their directions only instead of magnitude. A parallelogram must be constructed to determine the magnitudes of F_1 and F_2)

(b) $F_1 \rightarrow F_2$, F_5 (where F_5 is the intermediate force along the dash line)
$F_5 \rightarrow F_3$, F_4

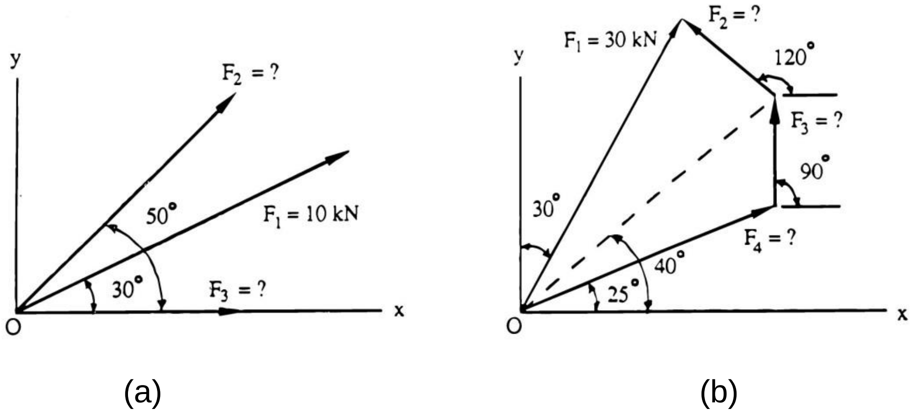


Fig. 4

Q.5

A force F and a moment M (around O_1) acting on the top flange of an I-shaped column section (310UC283) are detailed in Fig. 5. Compute the moment with respect to point O.

Hint: *Resolve the inclined force into x and y components.*

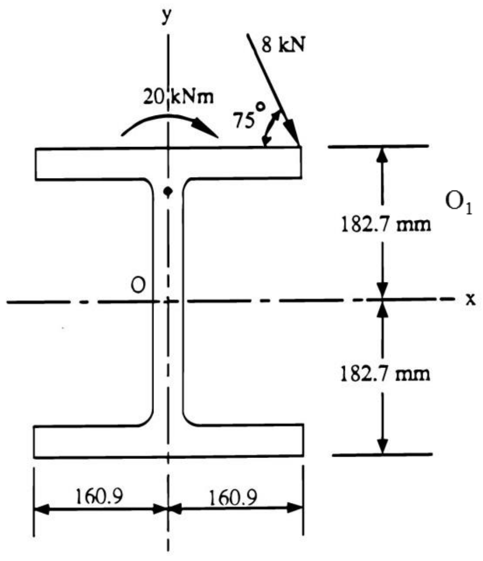


Fig. 5

EQUILIBRIUM EQUATIONS

Q.1
An arch structure is subjected to a force and a moment, as shown in Fig. 1. Set up, quantitatively, the relevant equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$. Note that the moment equations should be with respect to point O as indicated. Also compute F_1, F_2 and F_3.

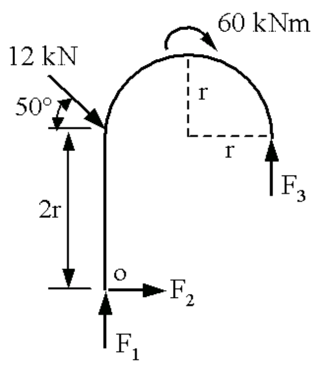


Fig. 1

Q.2
For each of the planar systems shown in Fig. 2, set up, quantitatively, the relevant equilibrium equations, $\sum Fx = 0$, $\sum Fy = 0$ and $\sum M = 0$. Note that the moment equations should be with respect to point O as indicated. Also compute F_1, F_2 and F_3.

Hint: Take the radius "r" as 2m.

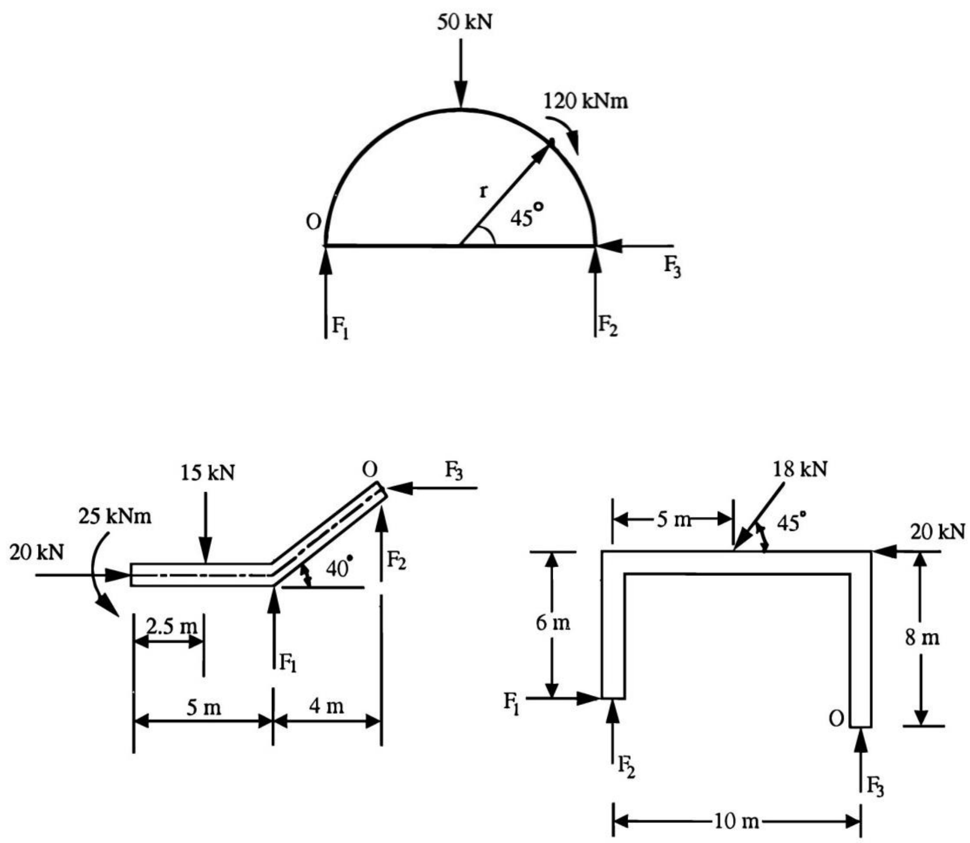


Fig. 2

SUPPORT REACTIONS AND FREE BODY DIAGRAMS

Q.1

Fig. 1 shows details of a simply-supported frame with overhang. Compute the support reactions and indicate their directions.

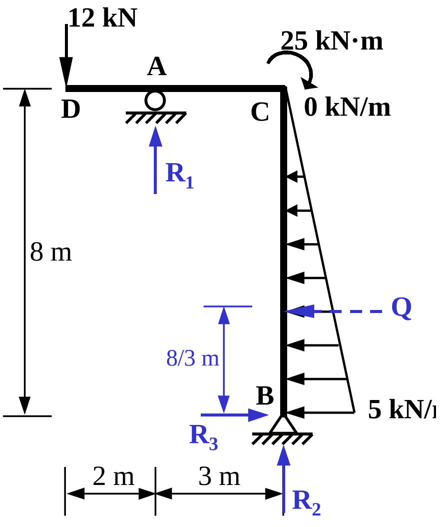


Fig. 1

Q.2

Fig. 2 details a cantilever frame. Compute the reactions at support **A** and indicate their directions.

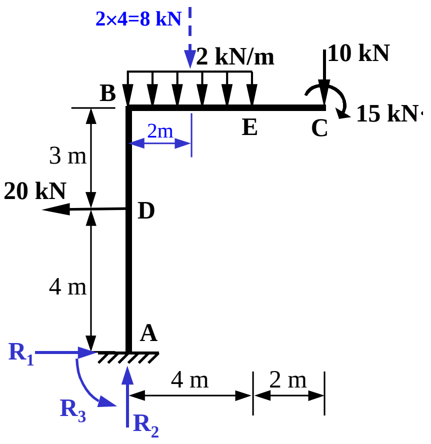


Fig. 2

Q.3

Fig. 3 illustrates a simply-supported frame subjected to the combination of a concentrated moment M (60 kN/m), a concentrated load Q (20 kN) and a linearly distributed load q (0 to 2.5 kN/m).

(a) Compute the reactions at supports **A** and **B**, and indicate their directions.

(b) (i) If the moment M is moved from point F to point C, would your solution to (a) be different?
Answer "yes" or "no" and explain why?

(ii) What if the horizontal load Q is transferred to point F?
Answer "yes" or "no" and explain why?

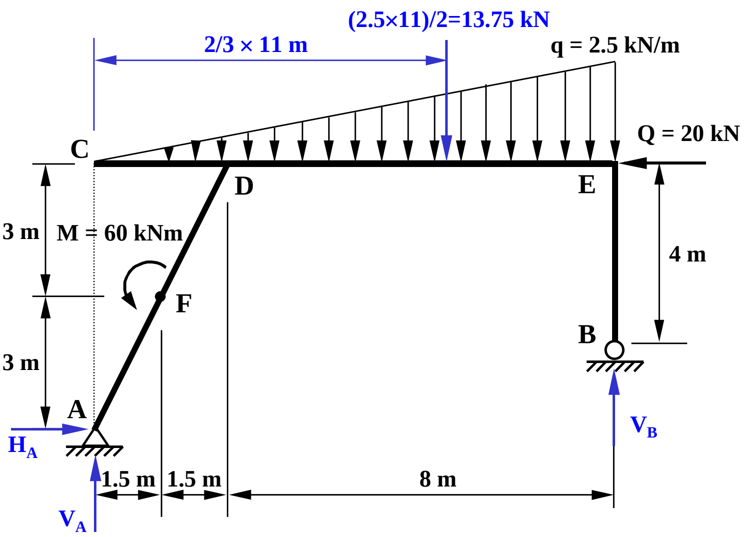


Fig. 3

Q.4

Consider each of the planar structures illustrated in Fig. 4 and determine if it is statically determinate, externally. If it is not, is it unstable or statically indeterminate?

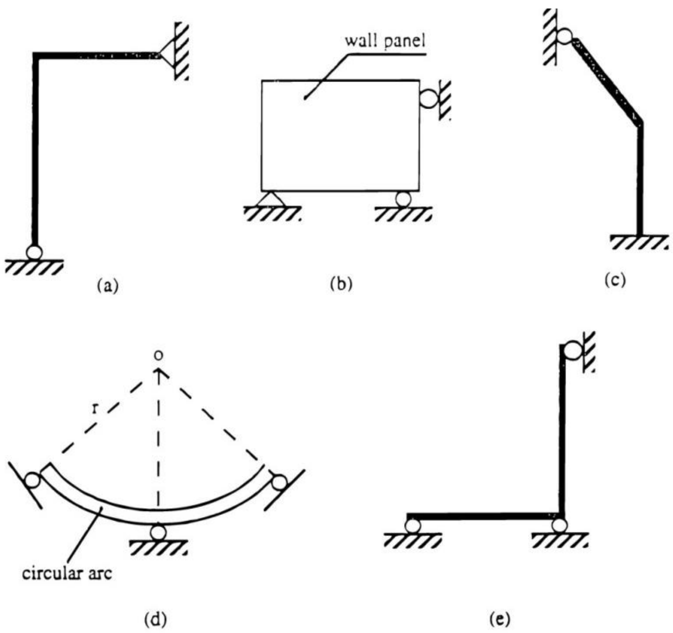


Fig. 4

Q.5

Fig. 5 shows details of the centre-line diagrams for a number of beams, trusses, arches and frames which are externally determinate. Compute the support reactions.

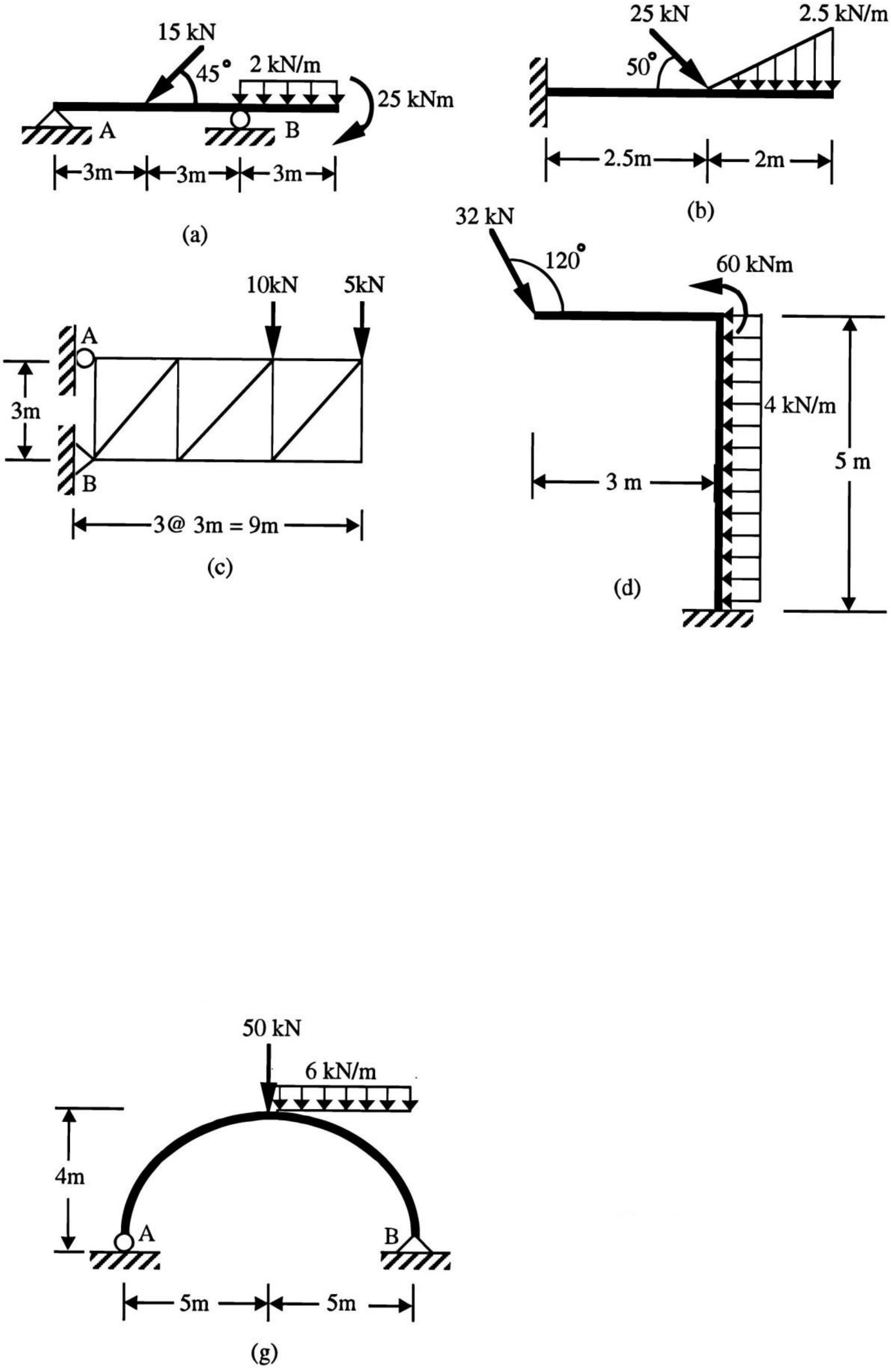


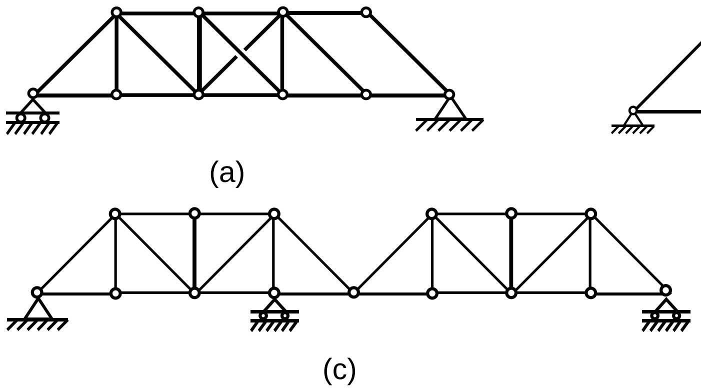
Fig. 5

ANALYSIS OF TRUSSES

Q.1

Inspect the “truss structures” illustrated in Fig. 1, make necessary calculations and answer, for each, the following questions:

(i) Statically determinate or indeterminate?
(ii) Stable or unstable? If unstable, why?

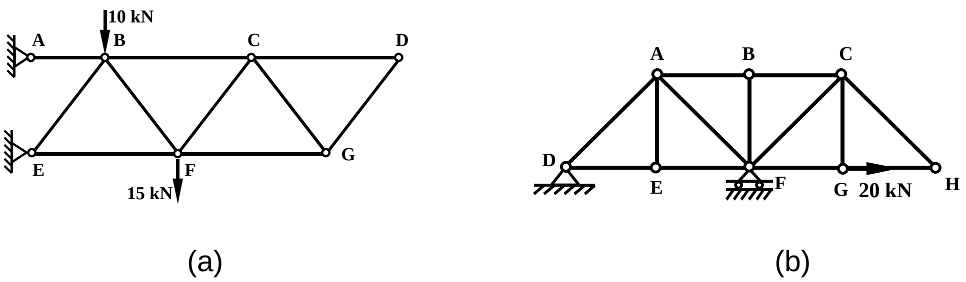


(a) (b) (c)

Fig. 1

Q.2

Identify the null members and null joints for the truss structures shown in Fig. 2.



(a) (b)

Fig. 2

Q.3

A truss is detailed in Fig. 3. For the given loading,
(a) Identify null member(s) and null joint(s).
(b) Determine the magnitude and nature of the axial forces in all members.

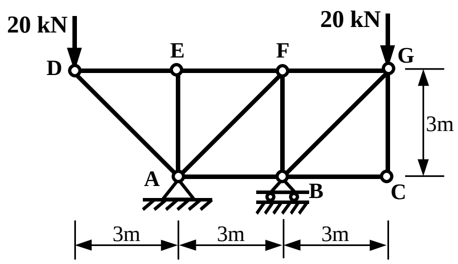


Fig. 3

Q.4

A truss is detailed in Fig. 4. The members HA and HE cross but are not connected to members GB and ID. For the given loading,

(a) Identify null member(s) and null joint(s).

(b) Determine the magnitude and nature of the axial forces in members **a**, **b** and **c**.

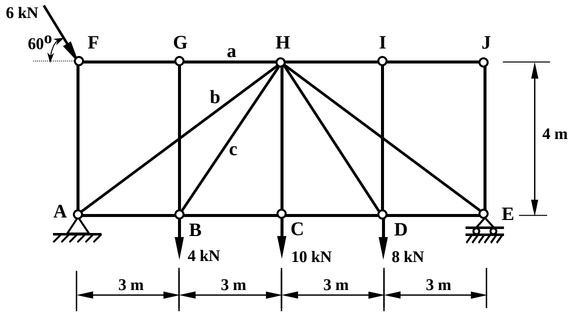


Fig. 4

Q.5

A truss is detailed in Fig. 5. For the given loading,

(a) Identify null member(s) and null joint(s).

(b) Determine the magnitude and nature of the axial force in members identified as **1**, **2**, **3** and **4**.

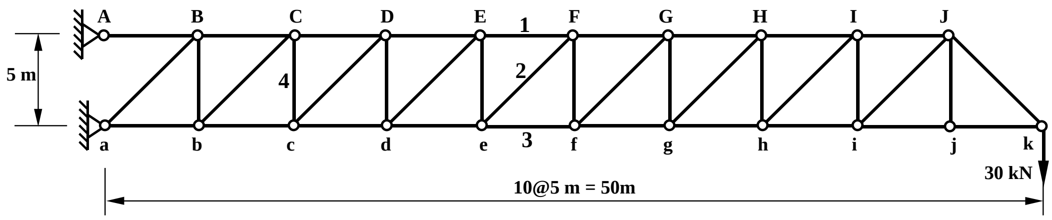


Fig. 5

Q.6

A truss is detailed in Fig. 6. For the given loading,

(a) Identify null member(s) and null joint(s).

(b) Determine the magnitude and nature of the axial force in members identified as **a**, **b** and **c**.

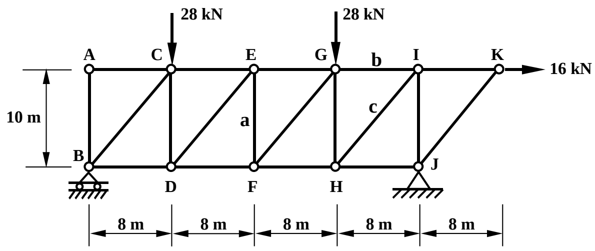


Fig. 6

Q.7 Consider each of the plane trusses shown in Fig. 7 and determine if it is statically determinate. If it is not, is it unstable or statically indeterminate?

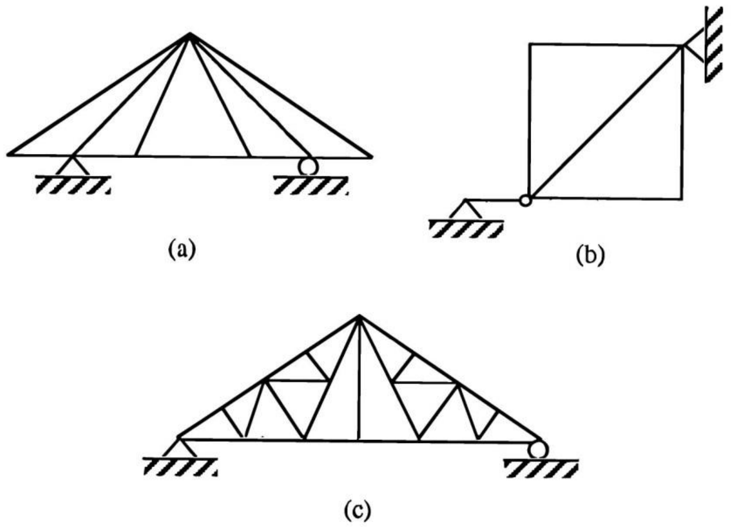


Fig. 7

Q.8 For each of the statically determinate trusses detailed in Fig. 8, perform a complete analysis of the member forces. Tabulate your results.

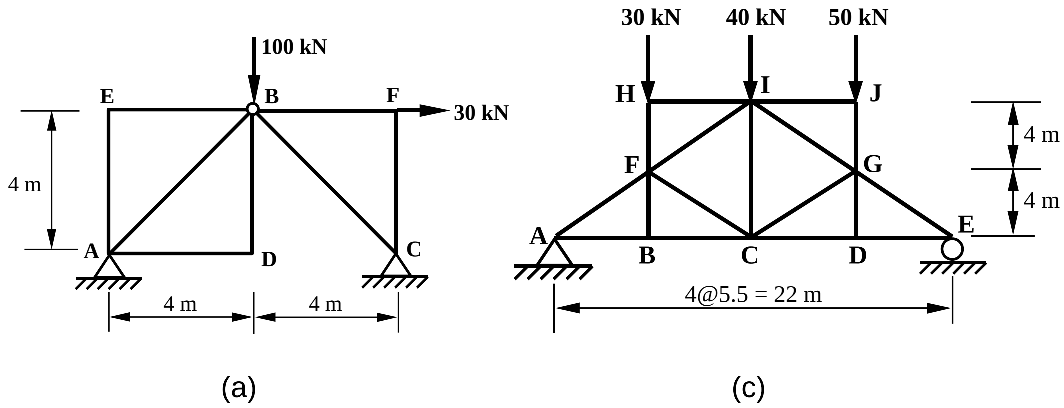


Fig. 8

Q.9 For each of the trusses illustrated in Fig. 9, determine the axial forces in the members identified as a, b and c.

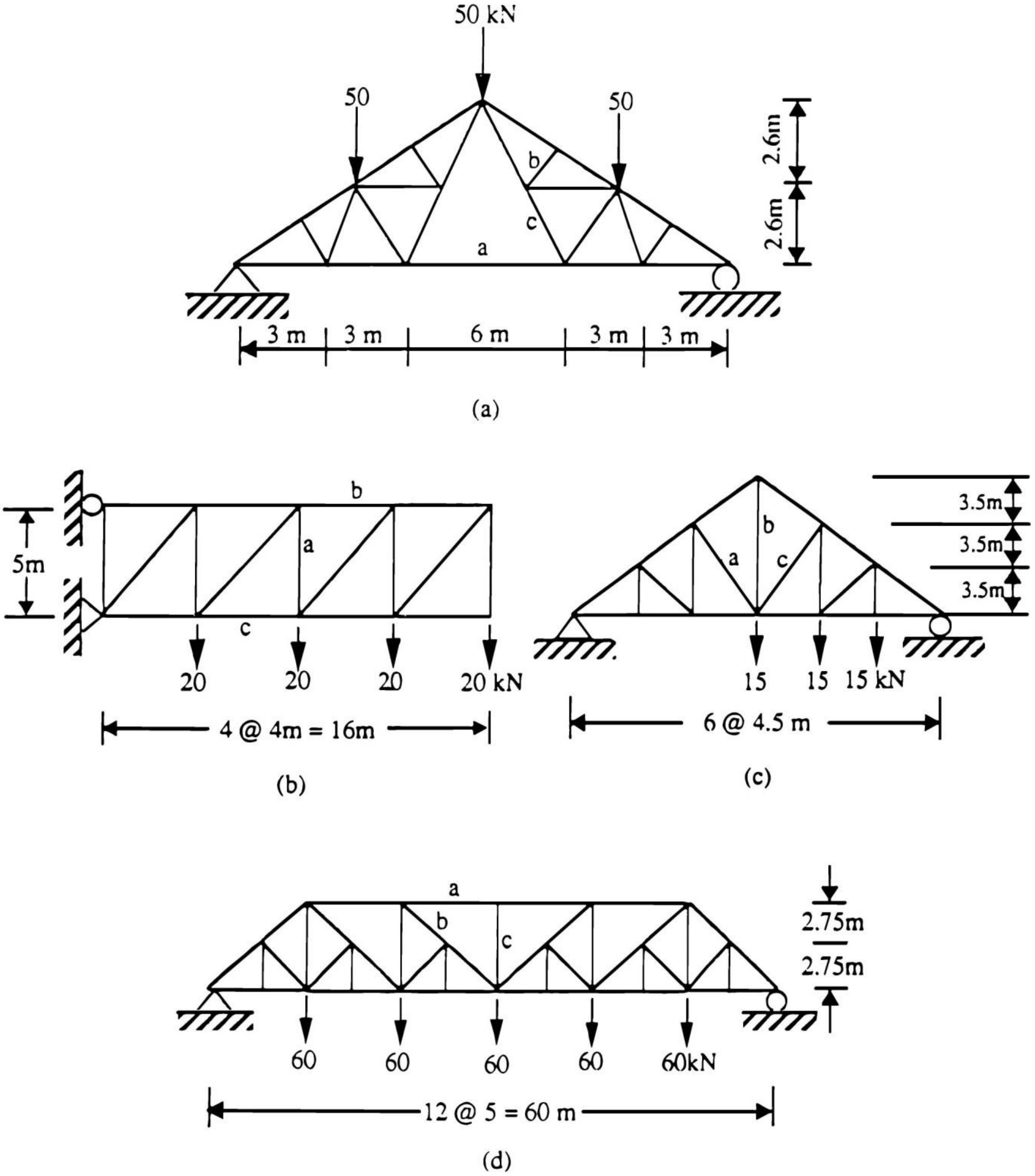


Fig. 9

FORCES AND MOMENTS IN SPACE

Q.1

A force F = 25 kN is applied to an eye bolt as shown in Fig. 1. Determine the magnitudes and directions of the components F_x, F_y and F_z parallel to the x, y and z axes, respectively.

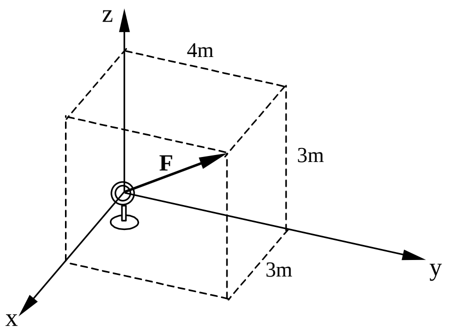


Fig. 1

Q.2

Fig. 2 below shows a straight cable, OB, in space. The tensile force in the cable is T = 100 kN. Determine the magnitudes of the components T_x, T_y, and T_z parallel respectively to the x, y and z axes.

Hint: Use ratios of length formulas.

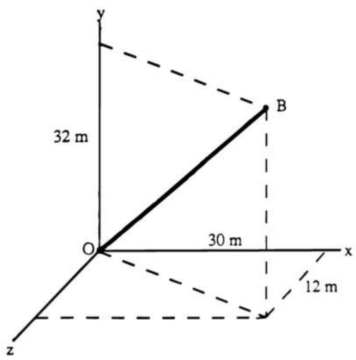


Fig. 2

Q.3

Three forces are applied to a cube as shown in Fig. 3. Set up, quantitatively, the relevant equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$, $\Sigma M_x = 0$, $\Sigma M_y = 0$ and $\Sigma M_z = 0$. Also compute F_1, F_2, F_3, M_1, M_2 and M_3.

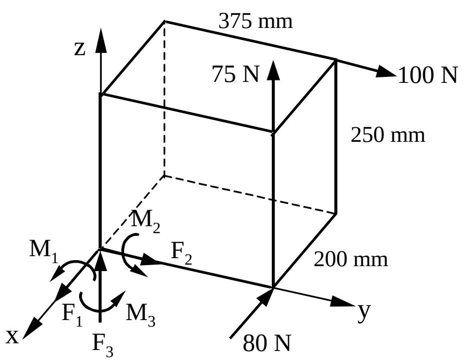
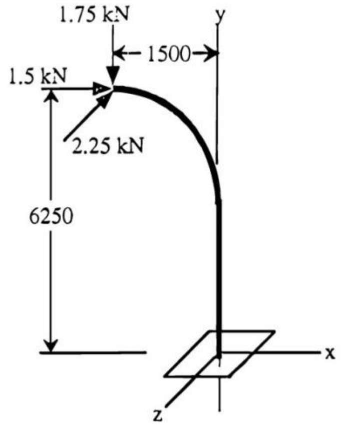


Fig. 3

Q.4

A lamp post is under the action of three concentrated loads, parallel respectively to the x, y, z, axes. Compute the reactions at the built-in support at the base. All dimensions are in mm.



Q.5

Fig. 5 shows a space truss with hinged supports at **E**, **F**, and **D**. Indicate if the truss is stable or not.

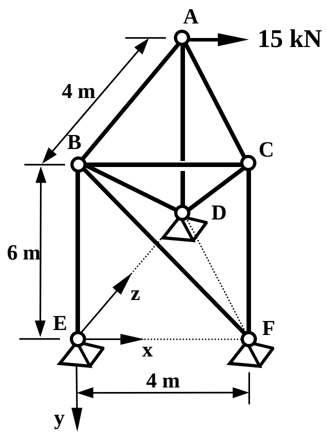


Fig. 5

Q.6

A frame structure subjected to a UDL in its own plane and a force in space at tip **C** is illustrated in Fig. 3. Determine the magnitude and direction of all support reactions at point **A**.

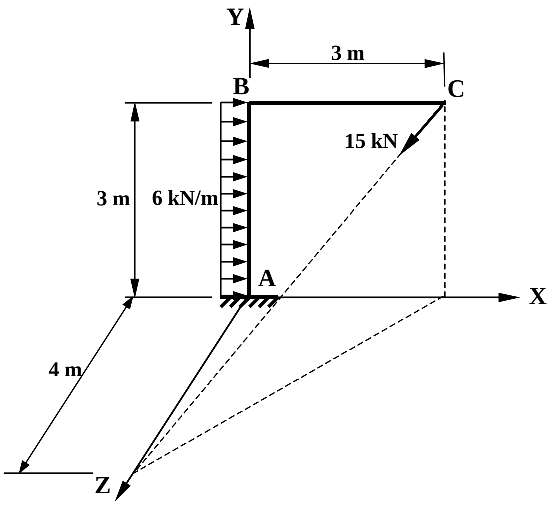


Fig. 6

CENTROIDS AND CENTRES OF GRAVITY

Q.1

For the composite area shown in Fig. 1, determine the location of the centroid.

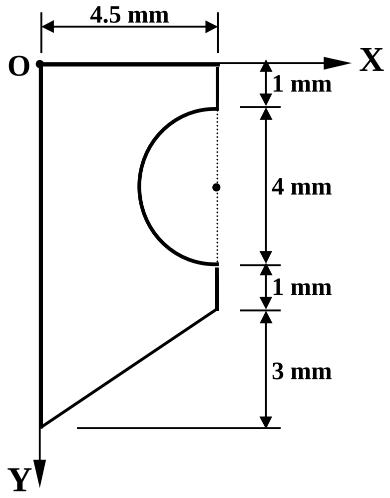


Fig. 1

Q.2

Some composite areas are depicted in Fig. 2. For each of these areas, determine the location of the centroid.

Hint: Use method of summation.

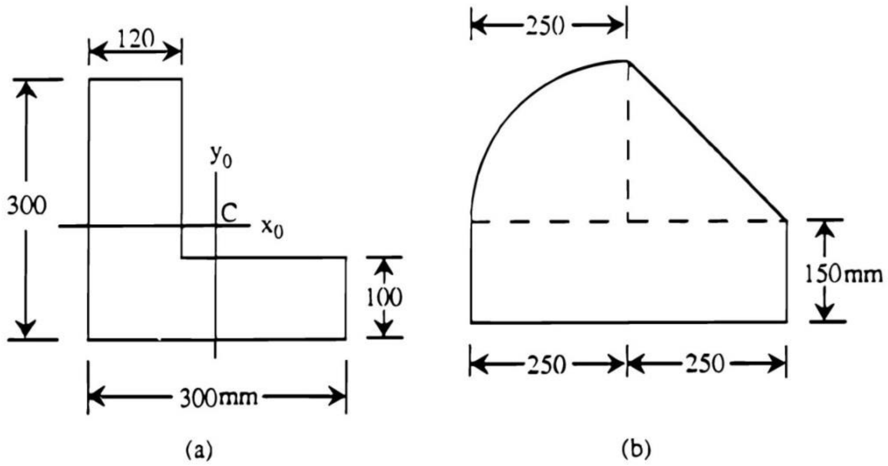


Fig. 2

MOMENT OF INERTIA

Q.1
A composite section is detailed in Fig. 1.
(a) Determine the location of the centroid "**C**" with respect to point **O**;
(b) Compute the moments of inertia with respect to the **X** and **Y** axes.

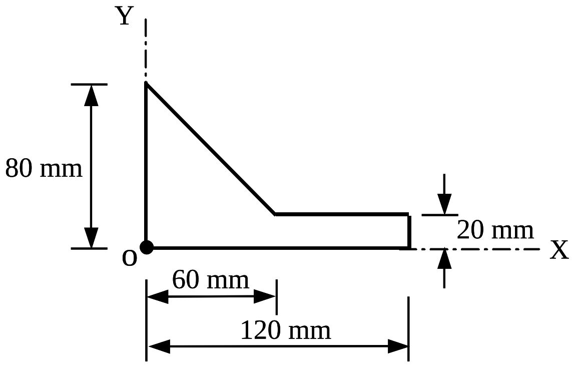


Fig. 1

Q.2
Compute the moments of inertia and the radii of gyration of the L-section shown in Fig. 2 with respect to the centroidal axes X_C and Y_C.

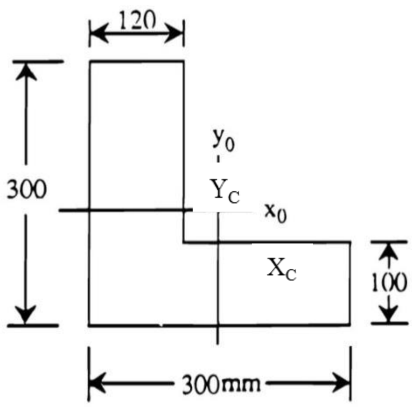


Fig. 2

Q.3
Determine the polar moment of inertia of the area illustrated in Fig. 3 with respect to (a) point P and (b) the centroid of the area.

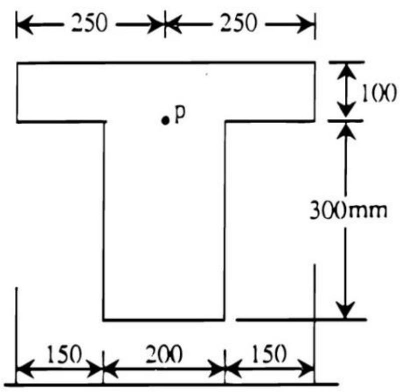


Fig. 3